Geom Dedicata (2013) 167:245–293
DOI 10.1007/s10711-012-9813-2

ORIGINAL PAPER

Elementary abelian 2-subgroups of compact Lie groups

Jun Yu

Received: 21 January 2012 / Accepted: 4 December 2012 / Published online: 18 December 2012

Abstract We classify elementary abelian 2-subgroups of compact simple Lie groups of adjoint type. This finishes the classification of elementary abelian p-subgroups of compact simple Lie groups (equivalently, complex linear algebraic simple groups) of adjoint type started in Griess (Geom Dedicata 39(3):253–305, 1991).

Keywords Elementary abelian 2-group · Automizer group · Involution type · Symplectic metric space · Translation subgroup

Mathematics Subject Classification (1991) 20E07 · 20E45 · 22C05

1 Introduction

For a positive integer m, let $C_m = \mathbb{Z}/m\mathbb{Z}$ be the cyclic group of order m. For a prime p and a positive integer n, an elementary abelian p-group of rank n is a finite group isomorphic to

$$(C_p)^n = \bigoplus_1^n C_p.$$

The goal of this paper is to study *elementary abelian p-subgroups* of compact simple Lie groups of adjoint type. Precisely, we focus on the case of $p = 2$. Here, we say a compact Lie group G is simple if its Lie algebra $\mathfrak{g}_0 = \mathrm{Lie}G$ is simple; and say it is of adjoint type if the adjoint homomorphism $\pi : G \longrightarrow \mathrm{Aut}(\mathfrak{g}_0)$ is an injective map. For a compact simple Lie algebra $\mathfrak{u}_0$ and any compact simple Lie group of adjoint type G with Lie algebra $\mathrm{Lie}G \cong \mathfrak{u}_0$, the adjoint homomorphism

$$\pi : G \longrightarrow \mathrm{Aut}(\mathfrak{u}_0)$$

J. Yu (✉)
Department of Mathematics, ETH Zürich, 8092 Zürich, Switzerland
e-mail: jun.yu@math.ethz.ch

Table 1 Torsion primes

$A_{n-1}, n \geq 2$	$B_n, n \geq 2$	$C_n, n \geq 3$	$D_n, n \geq 5$	D_4	E_6	E_7	E_8	F_4	G_2
$p\|2n$	2	2	2	2, 3	2, 3	2, 3	2, 3, 5	2, 3	2

is injective, so it suffices to study elementary abelian 2-subgroups of the compact Lie group $G = \mathrm{Aut}(\mathfrak{u}_0)$.

The structure of elementary abelian p-subgroups of a compact group G is related to the topology of G and its classifying space (cf. [2,4,10]). In the 1950s, Borel made an observation that, for a compact connected Lie group G, the cohomology ring $H^*(G,\mathbb{Z})$ has non-trivial p-torsion if and only if G has a non-toral elementary abelian p-subgroup (cf. [4,10]). Recall that, a subgroup of a compact Lie group G is called *toral* if it is contained in a maximal torus of G, otherwise it is called non-toral (cf. [10]). We call a prime p a torsion prime of a compact (not necessary connected) Lie group G if G has a non-toral elementary abelian p-subgroup. This definition is a bit different with that in [10]. For $G = \mathrm{Aut}(\mathfrak{u}_0)$ (the automorphism group of $\mathfrak{u}_0$) with $\mathfrak{u}_0$ a compact simple Lie algebra, the torsion primes are as in Table 1. From Table 1 we see that: the prime 2 is a torsion prime of $\mathrm{Aut}(\mathfrak{u}_0)$ for any compact simple Lie algebra $\mathfrak{u}_0$; when $\mathfrak{u}_0$ is a compact simple exceptional Lie algebra, any prime $p > 5$ is not a torsion prime and 5 is a torsion prime only when $\mathfrak{u}_0$ is of type $\mathbf{E_8}$.

The study of elementary abelian p-subgroups began at 1950s (or even earlier) by the famous mathematicians Borel, Serre, et al. In the 1990s, Griess [5] got a classification of maximal elementary abelian p-subgroups of linear algebraic simple groups (of adjoint type) defined over an algebraic closed field of characteristic 0. Since there exists a one-one correspondence between conjugacy classes of compact subgroups of a complex semisimple Lie algebraic group and such subgroups of (any of) its maximal compact subgroup (cf. Appendix of [1]), so we also have a classification of maximal elementary abelian p-subgroups of compact simple Lie groups of adjoint type. For odd primes p, non-toral elementary abelian p-subgroups are more or less well understood from [5] and [2]. Precisely, when $\mathfrak{u}_0$ is a compact exceptional simple Lie algebra, non-toral elementary abelian p-subgroups of $\mathrm{Aut}(\mathfrak{u}_0)$ are classified up to conjugacy. When $p > 5$, such subgroups don't exist; when $p = 5$, there is a unique conjugacy class in $\mathrm{Aut}(\mathfrak{e}_8)$ (cf. [5]); when $p = 3$, there are some conjugacy classes when $\mathfrak{u}_0$ is of type E_6, E_7, E_8 or F_4 (cf. [2,5]). But a complete classification is impossible when $\mathfrak{u}_0$ is a classical simple Lie algebra, since some complicated combinatorial problem will arise.

In this paper, we will first study elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{u}_0)$ for compact classical simple Lie algebras $\mathfrak{u}_0$ systematically. The method is to define and use linear algebraic structures on them (a bilinear form m or a bilinear form m together with a function μ, all with values in $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$). For any compact exceptional simple Lie algebra $\mathfrak{u}_0$, we classify elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{u}_0)$ up to conjugation and calculate their automizer groups (cf. Definition 3.4). A simple account of this classification is as follows. Theorem 1.1 follows by combining Corollaries 4.2, 5.3, 6.8, 7.24 and 8.13.

Theorem 1.1 *For* $\mathfrak{u}_0 = \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$, *there are* 51, 78, 66, 12, 4 *conjugacy classes of elementary abelian* 2*-subgroups in* $\mathrm{Aut}(\mathfrak{u}_0)$ *respectively.*

This paper is organized as follows. In Sect. 2, we do the classification for classical simple Lie algebras, which amounts to classify elementary abelian 2-subgroups of the groups $\mathrm{PU}(n) \rtimes \langle\tau_0\rangle$, $\mathrm{O}(n)/\langle -I\rangle$, $\mathrm{Sp}(n)/\langle -I\rangle$. Here, $\mathrm{PU}(n) = \mathrm{U}(n)/Z_n$ ($Z_n = \{\lambda I_n : |\lambda| = 1\}$)

is the projective unitary group and $\tau_0 =$ complex conjugation. We have $\tau_0^2 = 1$ and

$$\tau_0[A]\tau_0^{-1} = [\overline{A}], \quad \forall A \in \mathrm{U}(n).$$

In the first case, we will separate the discussion of subgroups contained in $\mathrm{PU}(n)$ and those not contained in it.

For an elementary abelian 2-subgroup F of $\mathrm{PU}(n)$, define a map

$$m : F \times F \longrightarrow \{\pm 1\}$$

by $m(x, y) = \lambda$ if $x = [A]$, $y = [B]$ and $ABA^{-1}B^{-1} = \lambda I$. We show that m is a bilinear form when F is viewed as a vector space over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ and $\{\pm 1\}$ is identified with $\mathbb{F}_2$. We also prove that $\ker m$ is diagonalizable and the conjugacy class of F is determined by the conjugacy class of $\ker m$ and the number $\mathrm{rank}(F/\ker m)$. This gives F a structure we called symplectic vector space. For an elementary abelian 2-subgroup F of $\mathrm{O}(n)/\langle -I\rangle$ or $\mathrm{Sp}(n)/\langle -I\rangle$, we define a bilinear map $m : F \times F \longrightarrow \{\pm 1\}$ and a function

$$\mu : F \longrightarrow \{\pm 1\}.$$

The definition of m is similar as in the $\mathrm{PU}(n)$ case; $\mu(x) = \lambda$ if $x = [A]$ and $A^2 = \lambda I$. The bilinear map m and the function μ satisfy a compatibility relation ($m(x, y) = \mu(x)\mu(y)\mu(xy)$). The compatible pair (m, μ) gives F a structure we called symplectic metric space and we get invariants r, s, ϵ, δ from the structure of a symplectic metric space. We show that the conjugacy class of F is determined by the conjugacy class of the subgroup $A_F = \ker(\mu|_{\ker m})$ and the numbers s, ϵ, δ. The consideration of elementary abelian 2-subgroups of the group $\mathrm{PU}(n) \rtimes \langle \tau_0\rangle$ is reduced to consideration of elementary abelian 2-subgroups of the above three groups.

In Sect. 2.4, we discuss a class of elementary abelian 2-subgroups of the groups $\mathrm{O}(n)/\langle -I\rangle$ and $\mathrm{Sp}(n)/\langle -I\rangle$ and introduce the notions of *symplectic vector space* and *symplectic metric space* and study their automorphism groups (Definition 3.4). They will play an important role in later sections.

In Sects. 4–8, we classify elementary abelian 2-subgroups of the automorphism group of any compact exceptional simple Lie algebra. A detailed account of the method is presented in Sect. 3. The study of some of these elementary abelian 2-subgroups is reduced to consideration of the class of subgroups of $\mathrm{Sp}(n)/\langle -I\rangle$ discussed in Sect. 2.4. Moreover, their automizer groups are described in terms of the automorphism groups of symplectic vector spaces or symplectic metric spaces.

Notation and conventions. Let $Z(G)$ ($\mathfrak{z}(\mathfrak{g})$) denote the center of a Lie group G (Lie algebra $\mathfrak{g}$) and G_0 denote the connected component of G containing identity element. For Lie groups $H \subset G$ (or Lie algebras $\mathfrak{h} \subset \mathfrak{g}$), let $C_G(H)$ ($C_{\mathfrak{g}}(\mathfrak{h})$) denote the centralizer of H in G ($\mathfrak{h}$ in $\mathfrak{g}$) and let $N_G(H)$ ($C_{\mathfrak{g}}(\mathfrak{h})$) denote the normalizer of H in G ($\mathfrak{h}$ in $\mathfrak{g}$). For an element x in G (or an automorphism of G), we also write G^x for the centralizer of x in G, so $G^x = C_G(x)$ when x is an element of G.

For any two elements $x, y \in G$, the notation $x \sim y$ means x, y are conjugate in G, i.e., $y = gxg^{-1}$ for some $g \in G$; and for a subgroup $H \subset G$, the notation $x \sim_H y$ means $y = gxg^{-1}$ for some $g \in H$. For two subsets $X_1, X_2 \subset G$, the notation $X_1 \sim X_2$ means $X_2 = gX_1g^{-1}$ for some $g \in G$; and for a subgroup $H \subset G$, the notation $X_1 \sim_H X_2$ means $X_2 = gX_1g^{-1}$ for some $g \in H$.

For a quotient group $G = H/N$, let $[x] = xN$ ($x \in H$) denote a coset.

All adjoint homomorphisms in this paper are denoted as π. This causes no ambiguity, as the reader can understand it is the adjoint homomorphism for which group everywhere π appears in this paper.

For a compact semisimple Lie algebra $\mathfrak{u}_0$, let $\mathrm{Aut}(\mathfrak{u}_0)$ be the group of automorphisms of $\mathfrak{u}_0$ and let $\mathrm{Int}(\mathfrak{u}_0)=\mathrm{Aut}(\mathfrak{u}_0)_0$. The elements in $\mathrm{Int}(\mathfrak{u}_0)$ are called inner automorphisms of $\mathfrak{u}_0$ and the elements in $\mathrm{Aut}(\mathfrak{u}_0)-\mathrm{Int}(\mathfrak{u}_0)$ are called outer automorphisms of $\mathfrak{u}_0$.

We denote by $\mathfrak{e}_6$ the compact simple Lie algebra of type $\mathbf{E_6}$. Let E_6 be the connected and simply connected Lie group with Lie algebra $\mathfrak{e}_6$. Let $\mathfrak{e}_6(\mathbb{C})$ and $\mathrm{E}_6(\mathbb{C})$ denote their complexifications. Similar notations will be used for other types. In the case of $G=\mathrm{E}_6$ or E_7, let c denote a non-trivial element in $Z(G)$. In the case of $\mathfrak{u}_0=\mathfrak{e}_7$, let

$$H_0'=\frac{H_2'+H_5'+H_7'}{2}\in i\mathfrak{e}_7\subset\mathfrak{e}_7(\mathbb{C})$$

(cf. Sect. 3.1).

Let $V=\mathbb{R}^n$ be an Euclidean linear space of dimension n with an orthogonal basis $\{e_1,e_2,\dots,e_n\}$ and $\mathrm{Pin}(n)$ ($\mathrm{Spin}(n)$) be the Pin (Spin) group of degree n associated to V. Write

$$c=e_1e_2\dots e_n\in\mathrm{Pin}(n).$$

Then c is in $\mathrm{Spin}(n)$ if and only if n is even, in this case $c\in Z(\mathrm{Spin}(n))$. If n is odd, then $\mathrm{Spin}(n)$ has a Spinor module M of dimension $2^{\frac{n-1}{2}}$. If n is even, then $\mathrm{Spin}(n)$ has two Spinor modules $M_+,\ M_-$ of dimension $2^{\frac{n-2}{2}}$. We distinguish M_+ and M_- by requiring that c acts on M_+ and M_- by scalar 1 and -1 respectively when $4|n$; and by $-i$ and i respectively when $4|n-2$.

For a prime p, let $\mathbb{F}_p=\mathbb{Z}/p\mathbb{Z}$ be the finite field with p elements. In particualr, for $p=2$, $\mathbb{F}_2=\mathbb{Z}/p\mathbb{Z}$ is a field with 2 elements. We have an isomorphism $\mathbb{F}_2\cong\{\pm1\}$ between the additive group $\mathbb{F}_2$ and the multiplicative group $\{\pm1\}$.

Let I_n be the $n\times n$ identity matrix. We define the following matrices,

$$I_{p,q}=\begin{pmatrix}-I_p&0\\0&I_q\end{pmatrix},\ J_n=\begin{pmatrix}0&I_n\\-I_n&0\end{pmatrix},\ \mathrm{J}_n'=\begin{pmatrix}0&\mathrm{I}_n\\\mathrm{I}_n&0\end{pmatrix},$$

$$I_{p,q}'=\begin{pmatrix}-I_p&0&0&0\\0&I_q&0&0\\0&0&-I_p&0\\0&0&0&I_q\end{pmatrix},$$

$$J_{p,q}=\begin{pmatrix}0&I_p&0&0\\-I_p&0&0&0\\0&0&0&I_q\\0&0&-I_q&0\end{pmatrix},$$

$$K_p=\begin{pmatrix}0&0&0&I_p\\0&0&-I_p&0\\0&I_p&0&0\\-I_p&0&0&0\end{pmatrix}.$$

And we define the following groups,

$$\begin{aligned}\mathrm{Z}_m&=\{\lambda I_m|\lambda^m=1\},\\ \mathrm{Z}'&=\{(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)|\epsilon_i=\pm1,\epsilon_1\epsilon_2\epsilon_3\epsilon_4=1\},\\ \Gamma_{p,q,r,s}&=\left\langle\begin{pmatrix}-I_p&0&0&0\\0&-I_q&0&0\\0&0&I_r&0\\0&0&0&I_s\end{pmatrix},\begin{pmatrix}-I_p&0&0&0\\0&I_q&0&0\\0&0&-I_r&0\\0&0&0&I_s\end{pmatrix}\right\rangle.\end{aligned}$$

2 Matrix groups

Let $\mathrm{M}_n(\mathbb{R})$, $\mathrm{M}_n(\mathbb{C})$, $\mathrm{M}_n(\mathbb{H})$ be the set of $n \times n$ matrices with entries in the field $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ respectively. Let

$$\begin{aligned}
&\mathrm{O}(n) = \{X \in \mathrm{M}_n(\mathbb{R}) | XX^t = \mathrm{I}\}, \quad \mathrm{SO}(n) = \{X \in \mathrm{O}(n) | \det X = 1\},\\
&\mathrm{U}(n) = \{X \in \mathrm{M}_n(\mathbb{C}) | XX^* = \mathrm{I}\}, \quad \mathrm{SU}(n) = \{X \in \mathrm{U}(n) | \det X = 1\},\\
&\mathrm{Sp}(n) = \{X \in \mathrm{M}_n(\mathbb{H}) | XX^* = \mathrm{I}\}.
\end{aligned}$$

Defined as sets in this way, $\mathrm{O}(n),\ \mathrm{SO}(n),\ \mathrm{U}(n),\ \mathrm{SU}(n),\ \mathrm{Sp}(n)$ are actually Lie groups, i.e., groups with a smooth manifold structure. Moreover, they are compact Lie groups, i.e., the underlying manifolds are compact. Also let

$$\mathrm{PO}(n),\ \mathrm{PSO}(n),\ \mathrm{PU}(n),\ \mathrm{PSU}(n)$$

be the quotients of the groups $\mathrm{O}(n),\ \mathrm{SO}(n),\ \mathrm{U}(n),\ \mathrm{SU}(n)$ modulo their centers (so $\mathrm{PU}(n) \cong \mathrm{PSU}(n)$, which is the projective unitary group). Let

$$\begin{aligned}
&\mathfrak{so}(n) = \{X \in M_n(\mathbb{R}) | X + X^t = 0\},\\
&\mathfrak{su}(n) = \{X \in M_n(\mathbb{C}) | X + X^* = 0, \mathrm{tr} X = 0\},\\
&\mathfrak{sp}(n) = \{X \in M_n(\mathbb{H}) | X + X^* = 0\},
\end{aligned}$$

where X^t denotes the transposition of a matrix X and X^* denotes the conjugate transposition of X. Then $\mathfrak{so}(n),\ \mathfrak{su}(n),\ \mathfrak{sp}(n)$ are Lie algebras of $\mathrm{SO}(n),\ \mathrm{SU}(n),\ \mathrm{Sp}(n)$ respectively. They represent all isomorphism classes of compact classical simple Lie algebras.

2.1 Projective unitary groups

Let $G = \mathrm{PU}(n) = \mathrm{U}(n)/\mathrm{Z}_n$. Then

$$G \cong \mathrm{Int}(\mathfrak{su}(n)).$$

Any involution $x \in G$ is of the form $x = [A],\ A \in \mathrm{U}(n)$ with $A^2 = I$. Then

$$A \sim I_{p,n-p} = \begin{pmatrix} -I_p & 0 \\ 0 & I_{n-p} \end{pmatrix}$$

for some $p,\ 1 \leq p \leq n-1$. One has

$$(\mathrm{U}(n)/\mathrm{Z}_n)^{[I_{p,n-p}]} = (\mathrm{U}(p) \times \mathrm{U}(n-p))/\mathrm{Z}_n \text{ if } p \neq \frac{n}{2}$$

and

$$(\mathrm{U}(n)/\mathrm{Z}_n)^{[I_{\frac{n}{2},\frac{n}{2}}]} = ((\mathrm{U}(n/2) \times \mathrm{U}(n/2))/\mathrm{Z}_n) \rtimes \langle [J'_n] \rangle.$$

Let $F \subset G$ be an elementary abelian 2-subgroup. For any $x, y \in F$, choose $A, B \in \mathrm{U}(n)$ with $A^2 = B^2 = I$ representing x, y (that is, $x = [A]$ and $y = [B]$). Then

$$1 = xyx^{-1}y^{-1} = (ABA^{-1}B^{-1})Z_n/Z_n \Longrightarrow [A, B] = \lambda_{A,B} I$$

for some $\lambda_{A,B} \in \mathbb{C}$. It is clear that $\lambda_{A,B} \in \mathbb{C}$ doesn't depend on the choice of A and B. Moreover, since $x^2 = y^2 = 1$, we have $\lambda_{A,B} = \pm 1$.

For any $x, y \in F$, define

$$m(x, y) = m_F(x, y) = \lambda_{A,B}.$$

Lemma 2.1 *For any $x, y, z \in F$, $m(x, x) = m(x, y)m(y, x) = 1$ and $m(xy, z) = m(x, z)m(y, z)$.*

Proof $m(x, x) = m(x, y)m(y, x) = 1$ is clear. Choose $A, B, C \in \mathrm{U}(n)$ with $A^2 = B^2 = C^2 = I$ representing x, y, z. Let $[A, C] = \lambda_1 I$, $[B, C] = \lambda_2 I$ for some numbers $\lambda_1, \lambda_2 = \pm 1$. We have

$$[AB, C] = A[B, C]A^{-1}[A, C] = A(\lambda_2 I)A^{-1}(\lambda_1 I) = (\lambda_1\lambda_2)I.$$

So $m(xy, z) = m(x, z)m(y, z)$. □

If we regard F as a vector space on $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ and identify $\{\pm 1\}$ with $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$, Lemma 2.1 just said m is an *anti-symmetric bilinear 2-form* on F. Let

$$\ker m = \{x \in F | m(x, y) = 1, \forall y \in F\}.$$

Then it is a subgroup of F.

For an even n, let $\Gamma_0 = \langle [I_{\frac{n}{2},\frac{n}{2}}], [J'_{\frac{n}{2}}] \rangle$, then any non-identity element of Γ_0 is conjugate to $I_{\frac{n}{2},\frac{n}{2}}$ and

$$(\mathrm{U}(n)/\mathrm{Z}_n)^{\Gamma_0} \cong (\mathrm{U}(n/2)/\mathrm{Z}_{\frac{n}{2}}) \times \Gamma_0.$$

Lemma 2.2 *For a Klein four subgroup $F \subset G$, if m_F is non-trivial, then F is conjugate to Γ_0.*

Proof Choose $A, B \in \mathrm{U}(n)$ with $A^2 = B^2 = I$ and $F = \langle [A], [B] \rangle$. Since m_F is non-trivial, we have $[A, B] = -I$. Since $A^2 = I$, we may assume that $A = I_{p,n-p}$ for some $1 \leq p \leq \frac{n}{2}$. From $[A, B] = -I$, we get $ABA^{-1} = -B$. Then B is of the form

$$B = \begin{pmatrix} & B_1 \\ B_2^t & \end{pmatrix}$$

for some $B_1, B_2 \in M_{p,n-p}$. Since B is invertible, we get $p = \frac{n}{2}$. Since $B^2 = I$, we have $B_1 B_2^t = I$. Let $S = \mathrm{diag}\{I_{n/2}, B_1\}$. Then

$$(SAS^{-1}, SBS^{-1}) = \left(\begin{pmatrix} -I_{\frac{n}{2}} & 0 \\ 0 & I_{\frac{n}{2}} \end{pmatrix}, \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ I_{\frac{n}{2}} & 0 \end{pmatrix} \right).$$

□

For any $m, k \geq 1$ and $A \in \mathrm{U}(m)$, let

$$\mathrm{D}(A) = \mathrm{diag}\{A, A, \ldots, A\}.$$

Then $D : \mathrm{U}(m) \longrightarrow \mathrm{U}(km)$ is the diagonal homomorphism.

Lemma 2.3 *For any two closed subgroups $S_1, S_2 \subset \mathrm{U}(m)$,*

$$\mathrm{D}(S_1) \sim \mathrm{D}(S_2) \Leftrightarrow S_1 \sim S_2.$$

Proof Since S_1, S_2 are closed subgroups of $\mathrm{U}(m)$, so they are compact groups. Then by character theory of representations of compact groups, both conditions in the lemma are equivalent to the existence of an isomorphism $\phi : S_1 \to S_2$ such that $\mathrm{tr}(\phi(x)) = \mathrm{tr}(x)$, $\forall x \in S_1$. Thus these two conditions are equivalent. □

Proposition 2.4 *Let F be an elementary abelian 2-subgroup of G,*

(1) *when* $\ker m = 1$, *the conjugacy class of F is determined by* $\mathrm{rank} F$*;*
(2) *in general,* $\ker m$ *is diagonalizable and the conjugacy class of F is determined by the conjugacy class of* $\ker m$ *and the number* $\mathrm{rank} F$.

Proof For (1), we prove by induction on n. Since $\ker m = 1$, so $\operatorname{rank} F$ is even. When $\operatorname{rank} F \geq 2$, choose any $x_1, x_2 \in F$ with $m(x_1, x_2) = -1$. By Lemma 2.2, we have

$$\langle x_1, x_2 \rangle \sim \Gamma_0.$$

We may and do assume that $\langle x_1, x_2 \rangle = \Gamma_0$, then

$$F \subset (\mathrm{U}(n)/\mathrm{Z}_n)^{\Gamma_0} = \Delta(\mathrm{U}(n/2)/\mathrm{Z}_{\frac{n}{2}}) \times \Gamma_0.$$

And so $F = \Delta(F') \times \Gamma_0$ for some $F' \subset \mathrm{U}(n/2)/\mathrm{Z}_{\frac{n}{2}}$. We also have $\ker m_{F'} = 1$. By induction, the conjugacy class of F' is determined by $\operatorname{rank} F'$, so the conjugcay class of F is determined by $\operatorname{rank} F$.

For (2), we have that $\pi^{-1}(\ker m)$ is abelian by the definition of m and $\ker m$, where π is the natural projection from $\mathrm{U}(n)$ to $\mathrm{U}(n)/\mathrm{Z}_n$. So $\pi^{-1}(\ker m)$ is diagonalizable. Then $\ker m$ is diagonalizable. We may write F as $F = \ker m \times F'$ with $m(\ker m, F') = 1$ and $m|_{F'}$ non-degenerate. By (1), the conjugcay class of F' is determined by

$$\operatorname{rank} F' = \operatorname{rank} F - \operatorname{rank}(\ker m).$$

Moreover, it is clear that

$$(\mathrm{U}(n)/\mathrm{Z}_n)^{F'} = \Delta(\mathrm{U}(n')/\mathrm{Z}_{n'}) \times F',$$

where $n' = n/2^{\frac{\operatorname{rank} F'}{2}}$. So $\ker m = \Delta(F'')$ for some $F'' \subset \mathrm{U}(n')/\mathrm{Z}_{n'}$. Fix F', by Lemma 2.3, the conjugacy class of $\ker m$ in $\mathrm{U}(n)/\mathrm{Z}_n$ and the conjugacy class of F'' in $\mathrm{U}(n')/\mathrm{Z}_{n'}$ determine each other. Since the conjugacy of F is determined by F' and the class of $\ker m$ in $(\mathrm{U}(n)/\mathrm{Z}_n)^{F'}$, we get the last statement of (2). □

2.2 Projective orthogonal and projective symplectic groups

Let $G = \mathrm{PO}(n) = \mathrm{O}(n)/\langle -I \rangle$, $n \geq 2$. Let F be an elementary abelian 2-subgroup of G. For any $x \in F$, choose $A \in \mathrm{O}(n)$ representing x, then $A^2 = \lambda_A I$ for some $\lambda_A = \pm 1$. For any $x, y \in F$, choose $A, B \in \mathrm{O}(n)$ representing x, y, then $[A, B] = \lambda_{A,B} I$ for some $\lambda_{A,B} = \pm 1$. The values of $\lambda_A, \lambda_{A,B}$ don't depend on the choice of A and B. For any $x, y \in F$, define

$$\mu(x) = \mu_F(x) = \lambda_A$$

and

$$m(x, y) = m_F(x, y) = \lambda_{A,B}.$$

Lemma 2.5 *For any* $x, y, z \in F$, $m(x, x) = 1$, $m(xy, z) = m(x, z)m(y, z)$, $\mu(1) = 1$ *and* $m(x, y) = \mu(x)\mu(y)\mu(xy)$.

Proof The equalities $m(x, x) = 1$ and $\mu(1) = 1$ are clear.

The proof for $m(xy, z) = m(x, z)m(y, z)$ is similar as that for 2.1.

Choose $A, B \in \mathrm{O}(n)$ representing x, y. Then

$$\begin{aligned}[A, B] &= ABA^{-1}B^{-1} \\ &= (AB)^2(B^2)^{-1}B(A^2)^{-1}B^{-1} \\ &= (\mu(xy)I)(\mu(y)I)^{-1}B(\mu(x)I)^{-1}B^{-2} \\ &= \mu(x)\mu(y)\mu(xyI).\end{aligned}$$

So $m(x, y) = \mu(x)\mu(y)\mu(xy)$. □

Lemma 2.6 *For an even n, $\mathfrak{su}(n)$ has two conjugacy classes of outer involutive automorphisms with representatives $\tau_0 =$ complex conjugation and $\tau_0' = \tau_0 \mathrm{Ad}(J_{n/2})$.*

For an odd n, $\mathfrak{su}(n)$ has a unique conjugacy class of outer involutive automorphisms with representative $\tau_0 =$ complex conjugation.

Proof This follows from Cartan 's classification of compact Riemannian symmetric pairs (cf. [7, Pages 451–455]). □

Lemma 2.7 *Let F be an elementary abelian 2-subgroup of G.*

For $x \in F$, $\mu(x) = -1$ if and only if $x \sim [J_{\frac{n}{2}}]$.

For $x, y \in F$ with $m(x, y) = -1$,

(1) *when $\mu(x) = \mu(y) = -1$, we have $(x, y) \sim ([J_{\frac{n}{2}}], [K_{\frac{n}{4}}])$;*
(2) *when $\mu(x) = \mu(y) = 1$, we have $(x, y) \sim ([I_{\frac{n}{2}, \frac{n}{2}}], [J'_{\frac{n}{2}}])$.*

Proof If $\mu(x) = -1$, then $x = [A]$ for some $A \in \mathrm{O}(n)$ with $A^2 = -I$. Then $A \sim J_{\frac{n}{2}}$, so $x \sim [J_{\frac{n}{2}}]$.

The proof of (2) is the same as that for Lemma 2.2.

For (1), first we may and do assume that $x = [J_{\frac{n}{2}}]$ by the first statement proved above. Then $\mathfrak{so}(n)^x = \mathfrak{u}(n/2)$. By Lemma 2.6, after replace y by some gyg^{-1} with $g \in G^x$, we may assume that

$$(\mathfrak{u}(n/2))^y = \mathfrak{so}(n/2) \text{ or } \mathfrak{sp}(n/4).$$

Then a little more argument shows that $y = [K_{\frac{n}{4}}]$. □

Definition 2.8 For an elementary abelian 2-groupsub $F \subset G$, define

$$A_F = \ker(\mu|_{\ker m})$$

and

$$\mathrm{defe} F = |\{x \in F : \mu(x) = 1\}| - |\{x \in F : \mu(x) = -1\}|.$$

We call $\mathrm{defe} F$ the defect index of F.

Define (ϵ_F, δ_F) as follows,

- when $\mu|_{\ker m} \neq 1$, define $(\epsilon_F, \delta_F) = (1, 0)$;
- when $\mu|_{\ker m} = 1$ and $\mathrm{defe} F < 0$, define $(\epsilon_F, \delta_F) = (0, 1)$;
- when $\mu|_{\ker m} = 1$ and $\mathrm{defe} F > 0$, define $(\epsilon_F, \delta_F) = (0, 0)$.

Define $r_F = \mathrm{rank} A_F$ and $s_F = \frac{1}{2}\mathrm{rank}(F/\ker m) - \delta_F$.

We will see in the proof of Proposition 2.12 that $\mathrm{defe} F = 0$ if and only if $\mu|_{\ker m} \neq 1$. It is clear that $\epsilon_F, \delta_F, r_F, s_F$ and the conjugcay class of A_F are determined by the conjugacy class of F.

Let $\Gamma_1 = \langle [I_{\frac{n}{2}, \frac{n}{2}}], [J'_{\frac{n}{2}}] \rangle$ and $\Gamma_2 = \langle [J_{\frac{n}{2}}], [K_{\frac{n}{4}} \rangle]$. Then $\mathrm{defe}\Gamma_1 = 2$, $\mathrm{defe}\Gamma_2 = -2$,

$$(\mathrm{O}(n)/\langle -I \rangle)^{\Gamma_1} = \Delta(\mathrm{O}(\frac{n}{2})/\langle -I \rangle) \times \Gamma_1$$

and

$$(\mathrm{O}(n)/\langle -I \rangle)^{\Gamma_2} = \Delta(\mathrm{Sp}(\frac{n}{4})/\langle -I \rangle) \times \Gamma_2.$$

Lemma 2.9 *Let F be a non-trivial elementary abelian 2-subgroup of $\mathrm{O}(n)/\langle -I \rangle$, if $\mathrm{rank}(F/\ker m) > 2$, then there exists a Klein four subgroup $F' \subset F$ such that $F' \sim \Gamma_1$.*

Proof Choose a subgroup $F'' \subset F$ such that $F = \ker m \times F''$, then $\ker(m_{F''}) = 1$ and $\operatorname{rank} F'' > 2$. Replace F by F'', we may assume that $\ker m = 1$ and $\operatorname{rank} F > 2$.

We first show that, there exists $1 \neq x \in F$ with $\mu(x) = 1$. From $\operatorname{rank} F > 2$, we get $\operatorname{rank} F \geq 4$ since it is even (m_F is non-degenerate). Suppose that any $1 \neq x \in F$ has $\mu(x) = -1$. Then for any distinct non-trivial elements $x, y \in F$, we have

$$m(x, y) = \mu(x)\mu(y)\mu(xy) = -1$$

by Lemma 2.5. This contradicts that m is bilinear on F.

Upon we get $1 \neq x \in F$ with $\mu(x) = 1$, choose any $z \in F$ with $m(x, z) = -1$. Then

$$\mu(xz)\mu(z) = m(x, z)\mu(x) = -1.$$

So exactly one of $\mu(z), \mu(xz)$ is equal to -1. By Lemma 2.7, we have $\langle x, z\rangle \sim \Gamma_1$. □

Lemma 2.10 *Let F be a non-trivial elementary abelian 2-subgroup of $\mathrm{O}(n)/\langle -I\rangle$. If* $\operatorname{rank}(\ker m/A_F) = 1$ *and* $\operatorname{rank}(F/A_F) > 1$*, then there exists a Klein four subgroup $F' \subset F$ with $F' \sim \Gamma_1$.*

Proof Choose a subgroup $F'' \subset F$ such that $F = A_F \times F''$, then $\operatorname{rank}(\ker(m_{F''})) = 1$, $A_{F''} = 1$ and $\operatorname{rank} F'' > 1$. Replace F by F'', we may assume that $A_F = 1$, $\operatorname{rank}(\ker m) = 1$ and $\operatorname{rank} F > 1$.

The subgroup F is of the form $F = \ker m \times F''$ with $m(\ker m, F'') = 1$, $\operatorname{rank} F'' \geq 2$, and $m_{F''}$ non-degenerate. When $\operatorname{rank} F'' > 2$ or $F'' \sim \Gamma_1$, there exists $F' \subset F'$ with $F' \sim \Gamma_1$ by Lemma 2.9. Otherwise $F'' \sim \Gamma_2$. Choose $x, y \in F''$ generating F'' and $1 \neq z \in \ker m$, then

$$F' = \langle xz, yz\rangle \sim \Gamma_1$$

since $(\mu(xz), \mu(yz), \mu(xy)) = (1, 1, -1)$. □

For any $n \geq 1$, let $T : \mathrm{O}(n) \hookrightarrow \mathrm{U}(n),\ T' : \mathrm{Sp}(n/2) \hookrightarrow \mathrm{U}(n))$ be the natural inclusions.

Lemma 2.11 *For any two closed subgroups $S_1, S_2 \subset \mathrm{O}(n)$ or $S'_1, S'_2 \subset \mathrm{Sp}(n/2)$*

$$T(S_1) \sim T(S_2) \Leftrightarrow S_1 \sim S_2$$

and

$$T'(S'_1) \sim T'(S'_2) \Leftrightarrow S'_1 \sim S'_2.$$

Proof These follow from [6] Theorem 2.3 and [1] Theorem 8.1. □

Proposition 2.12 *Let F be an elementary abelian 2-subgroup of* $\mathrm{O}(n)/\langle -I\rangle$*,*

(1) *when* $\ker m = 1$*, the conjugacy class of F is determined by δ_F and s_F;*
(2) *in general,* $\ker m$ *is diagonalizable and the conjugacy class of F is determined by the conjugacy class of A_F and the invariants $(\epsilon_F, \delta_F, s_F)$.*
(3) *we have* $\operatorname{defe}(F) = (1 - \epsilon_F)(-1)^{\delta_F} 2^{r_F + s_F + \delta_F}$.

Proof For (1), since $\ker m = 1$, so $\operatorname{rank} F$ is even. When $\operatorname{rank} F = 2$, $F \sim \Gamma_1$ or Γ_2 by Lemma 2.2. When $\operatorname{rank} F \geq 2$, there exists a Klein four subgroup $F' \subset F$ with $F' \sim \Gamma_1$ by Lemma 2.9. We may and do assume that $\Gamma_1 \subset F$. Then

$$F \subset (\mathrm{O}(n)/\langle -I\rangle)^{\Gamma_1} = \Delta\left(\mathrm{O}(\frac{n}{2})/\langle -I\rangle\right) \times \Gamma_1.$$

So $F=\Delta(F')\times\Gamma_1$ for some $F'\subset \mathrm{O}(\frac{n}{2})/\langle -I\rangle$. By induction, we can show $\mathrm{defe} F\neq 0$ and the conjugacy class of F is determined by δ_F and s_F.

For (2), $\ker m$ is diagonalizable since $\pi^{-1}(\ker m)$ is abelian by the definition of m, where π is the natural projection

$$\pi:\mathrm{O}(n)\longrightarrow \mathrm{O}(n)/\langle -I\rangle.$$

We break the proof into two parts according to the value of ϵ_F. When $\epsilon_F=1$, by Lemma 2.10, F is of the form $F=\ker m\times F'$ with $m_{F'}$ non-degenerate and $\mathrm{defe} F'>0$. By (1), the conjugacy class of F' is determined by $s_F=\frac{\mathrm{rank} F'}{2}$. We have

$$(\mathrm{O}(n)/\langle -I\rangle)^{F'}=\Delta(\mathrm{O}(n')/\langle -I\rangle)\times F',$$

where $n'=\frac{n}{2^{\frac{\mathrm{rank} F'}{2}}}$. Fixing F', by Lemmas 2.3 and 2.11, the conjugacy class of $\ker m$ in $\mathrm{O}(n)/\langle -I\rangle$ determines the conjugacy class of it in $(\mathrm{O}(n)/\langle -I\rangle)^{F'}$. Moreover, as $\epsilon_F=1$ is given, the conjugacy class of $\ker m$ is determined by the conjugacy class of $A_F=\ker \mu|_{\ker m}$. So the conjugacy class of F is determined by that of A_F and the invariants (δ_F,s_F). When $\epsilon_F=0$, it is similar as the above proof for $\epsilon_F=1$ case to show that the conjugacy class of F is determined by the conjugacy class of A_F and the invariants (δ_F,s_F).

(3) follows from Lemma 2.7 and (2). □

The classification of elementary abelian 2-subgroup of $\mathrm{Sp}(n)/\langle -I\rangle$ is similar as that of $\mathrm{O}(n)/\langle -I\rangle$. We give the definitions and results below but omit the proofs.

Let F be an elementary abelian 2 subgroup of $\mathrm{Sp}(n)/\langle -I\rangle$, $n\geq 2$. For any $x\in F$, choose $A\in\mathrm{Sp}(n)$ representing x, then $A^2=\lambda_A I$ for some $\lambda_A=\pm 1$. For any $x,y\in F$, choose $A,B\in\mathrm{Sp}(n)$ representing x,y, then $[A,B]=\lambda_{A,B}I$ for some $\lambda_{A,B}=\pm 1$. The values of $\lambda_A,\lambda_{A,B}$ don't depend on the choice of A,B. For any $x,y\in F$, define

$$\mu(x)=\mu_F(x)=\lambda_A$$

and

$$m(x,y)=m_F(x,y)=\lambda_{A,B}.$$

Lemma 2.13 *Let F be an elementary abelian 2-subgroup of* $\mathrm{Sp}(n)/\langle -I\rangle$. *For any $x,y,z\in F$, $m(x,x)=1$, $m(xy,z)=m(x,z)m(y,z)$, $\mu(1)=1$ and*

$$m(x,y)=\mu(x)\mu(y)\mu(xy).$$

Lemma 2.14 *Let F be an elementary abelian 2-subgroup of* $\mathrm{Sp}(n)/\langle -I\rangle$.

For $x\in F$, $\mu(x)=-1$ if and only if $x\sim[J_{\frac{n}{2}}]$.

For $x,y\in F$ with $m(x,y)=-1$,

(1) *when $\mu(x)=\mu(y)=-1$, we have $(x,y)\sim([\mathbf{i}I],[\mathbf{j}I])$;*
(2) *when $\mu(x)=\mu(y)=1$, we have $(x,y)\sim([I_{\frac{n}{2},\frac{n}{2}}],[J'_{\frac{n}{2}}])$.*

Definition 2.15 For an elementary abelian 2-subgroup $F\subset\mathrm{Sp}(n)/\langle -I\rangle$, define

$$A_F=\ker(\mu|_{\ker m})$$

and the defect index

$$\mathrm{defe} F=|\{x\in F:\mu(x)=1\}|-|\{x\in F:\mu(x)=-1\}|.$$

Define (ϵ_F,δ_F) as follows,

- when $\mu|_{\ker m} \neq 1$, define $(\epsilon_F, \delta_F) = (1, 0)$;
- when $\mu|_{\ker m} = 1$ and $\mathrm{defe} F < 0$, define $(\epsilon_F, \delta_F) = (0, 1)$;
- when $\mu|_{\ker m} = 1$ and $\mathrm{defe} F > 0$, define $(\epsilon_F, \delta_F) = (0, 0)$.

Define $r_F = \mathrm{rank} A_F$ and $s_F = \frac{1}{2}\mathrm{rank}(F/\ker m) - \delta_F$.

Proposition 2.16 *Let F be an elementary abelian 2-subgroup of* $\mathrm{Sp}(n)/\langle -I\rangle$*,*

(1) *when* $\ker m = 1$*, the conjugacy class of F is determined by δ_F and s_F;*
(2) *in general,* $\ker m$ *is diagonalizable and the conjugacy class of F is determined by the conjugacy class of A_F and the invariants $(\epsilon_F, \delta_F, s_F)$.*
(3) *we have* $\mathrm{defe}(F) = (1-\epsilon_F)(-1)^{\delta_F} 2^{r_F+s_F+\delta_F}$.

2.3 Twisted projective unitary groups

For $n \geq 3$, let $G = \mathrm{Aut}(\mathfrak{su}(n))$, which has two connected components and $G_0 = \mathrm{Int}(\mathfrak{su}(n)) = \mathrm{PU}(n) = \mathrm{U}(n)/\mathrm{Z}_n$. When n is even, G has two conjugacy classes of outer involutions with representatives $\tau_0 = \textit{complex conjugation}$ and $\tau_0\mathrm{Ad}(J_{n/2})$; when n is odd, G has a unique conjugacy class of outer involutions with representative τ_0. We have (cf. [8, Table 2])

$$\mathrm{Int}(\mathfrak{su}(n))^{\tau_0} = \mathrm{O}(n)/\langle -I\rangle$$

and

$$\mathrm{Int}(\mathfrak{su}(n))^{\tau_0\mathrm{Ad}(J_{n/2})} = \mathrm{Sp}(n/2)/\langle -I\rangle.$$

Let F be an elementary abelian 2-subgroup of G. For the subgroup $F \cap \mathrm{Int}(\mathfrak{su}(n))$ of $\mathrm{Int}(\mathfrak{su}(n)) = \mathrm{PU}(n)$, we have a bilinear form

$$m : F \cap \mathrm{Int}(\mathfrak{su}(n)) \times F \cap \mathrm{Int}(\mathfrak{su}(n)) \longrightarrow \{\pm 1\}.$$

Moreover, we define a function

$$\mu : F - F \cap \mathrm{Int}(\mathfrak{su}(n)) \longrightarrow \{\pm 1\}$$

by $\mu(z) = 1$ if $z \sim \tau_0$, and $\mu(z) = -1$ if $z \sim \tau_0\mathrm{Ad}(J_{n/2})$. On the other hand, for any $z \in F - \mathrm{Int}(\mathfrak{su})(n)$, define $\mu_z : F \cap \mathrm{Int}(\mathfrak{su}(n)) \longrightarrow \{\pm 1\}$ and

$$m_z : (F \cap \mathrm{Int}(\mathfrak{su}(n))) \times (F \cap \mathrm{Int}(\mathfrak{su}(n))) \longrightarrow \{\pm 1\}$$

from the inclusion

$$F \cap \mathrm{Int}(\mathfrak{su}(n)) \subset \mathrm{Int}(\mathfrak{su}(n))^z \cong \mathrm{O}(n)/\langle -I\rangle \text{ or } \mathrm{Sp}(n/2)/\langle -I\rangle.$$

Definition 2.17 For an elementary abelian 2-subgroup $F \subset \mathrm{Aut}(\mathfrak{su}(n))$, define

$$A_F = \{x \in F \cap \mathrm{Int}(\mathfrak{su}(n)) | z \sim zx, \forall z \in F - F \cap \mathrm{Int}(\mathfrak{su}(n))\}$$

and the defect index

$$\mathrm{defe} F = |\{x \in F : x \sim \tau_0\}| - |\{x \in F : x \sim \tau_0\mathrm{Ad}(J_{n/2})\}|.$$

Define (ϵ_F, δ_F) as follows,

- when $\mathrm{defe} F = 0$, define $(\epsilon_F, \delta_F) = (1, 0)$;
- when $\mathrm{defe} F > 0$, define $(\epsilon_F, \delta_F) = (0, 0)$;
- when $\mathrm{defe} F < 0$, define $(\epsilon_F, \delta_F) = (0, 1)$.

Define $r_F = \mathrm{rank} A_F$ and $s_F = \frac{1}{2}\mathrm{rank}(F/\ker m) - \delta_F$.

Lemma 2.18 *Let F be an elementary abelian 2-subgroup of* $\mathrm{Aut}(\mathfrak{su}(n))$. *Then for any* $z \in F - F \cap \mathrm{Int}(\mathfrak{su}(n))$, *we have* $m_z = m$ *on* $F \cap \mathrm{Int}(\mathfrak{su}(n))$.

Proof For $z \in F - \mathrm{Int}(\mathfrak{su}(n))$ and $x, y \in F \cap \mathrm{Int}(\mathfrak{su}(n))$, by Lemma 2.2, 2.7 and 2.14, we have

$$m(x, y) = -1 \Leftrightarrow \langle x, y\rangle \sim \langle [I_{\frac{n}{2},\frac{n}{2}}], [J'_{\frac{n}{2}}]\rangle \Leftrightarrow m_z(x, y) = -1.$$

So $m_z(x, y) = m(x, y)$. □

Lemma 2.19 *For any* $z \in F - F \cap \mathrm{Int}(\mathfrak{su}(n))$ *and* $x \in F \cap \mathrm{Int}(\mathfrak{su}(n))$, $\mu_z(x) = \mu(z)\mu(zx)$.

Proof We may and do assume that $z = \tau_0$ or $\tau_0 \mathrm{Ad}(J_{n/2})$.

In the case of $z = \tau_0$ and $\mu_z(x) = 1$, we may assume that $x = [I_{p,n-p}] \in \mathrm{O}(n)/\langle -I\rangle = \mathrm{Int}(\mathfrak{su}(n))^z$ for some $0 \le p \le n$. Let $u = [\mathrm{diag}\{iI_p, I_{n-p}\}]$, then

$$\begin{aligned} uzu^{-1} &= z(z^{-1}uz)u^{-1} = z(\overline{u})u^{-1} \\ &= z[\mathrm{diag}\{-iI_p, I_{n-p}\}][\mathrm{diag}\{-iI_p, I_{n-p}\}] \\ &= z[I_{p,n-p}] = zx. \end{aligned}$$

So $zx \sim z$. And so $1 = \mu_z(x) = \mu(z)\mu(zx)$.

In the case of $z = \tau_0$ and $\mu_z(x) = -1$, we may assume that $x = [J_{n/2}] \in \mathrm{O}(n)/\langle -I\rangle = \mathrm{Int}(\mathfrak{su}(n))^z$. Then $zx = \tau_0 \mathrm{Ad}(J_{n/2}) = \tau'_0$. So $-1 = \mu_z(x) = \mu(z)\mu(zx)$.

The proof in the case of $z = \tau'_0 = \tau_0 \mathrm{Ad}(J_{n/2})$ is similar. □

Lemma 2.20 *For any elementary abelian 2-subgroup* $F \subset \mathrm{Aut}(\mathfrak{su}(n))$, *we have* $A_F \subset \ker m$ *and* $A_F = \ker(\mu_z|_{\ker m})$ *for any* $z \in F - F \cap \mathrm{Int}(\mathfrak{su}(n))$.

Proof Choose any $x \in A_F$ and choose an element $z \in F - F \cap \mathrm{Int}(\mathfrak{su}(n))$. By the definition of A_F, for any $y \in F \cap \mathrm{Int}(\mathfrak{su}(n))$, we have $\mu(zy) = \mu(zyx)$. In particular for $y = 1$, we have $\mu(z) = \mu(zx)$. Then

$$m(x, y) = m_z(x, y) = \mu_z(x)\mu_z(y)\mu_z(xy) = \mu(z)\mu(zx)\mu(zy)\mu(zxy) = 1.$$

So $A_F \subset \ker m$.

On the other hand, for any $x \in \ker m = \ker m_z$, $x \in A_F$ if and only if $\forall y \in F \cap \mathrm{Int}(\mathfrak{su}(n))$, $\mu(zy) = \mu(zyx)$. Since

$$\mu(zy)\mu(zyx) = \mu_z(y)\mu_z(xy) = m_z(x, y)\mu_z(x) = \mu_z(x),$$

we get that $A_F = \ker(\mu_z|_{\ker m})$. □

Proposition 2.21 *For an elementary abelian 2-subgroup F of* $\mathrm{Aut}(\mathfrak{su}(n))$ *which is not contained in* $\mathrm{Int}(\mathfrak{su}(n))$, $\ker m$ *is diagonalizable and the conjugacy class of F is determined by the conjugacy class of A_F and the invariants* $(\epsilon_F, \delta_F, \mathrm{rank} F)$.

Proof We break the proof into two cases.

When there exists $z \in F$ with $z \sim \tau_0$, we may and do assumen that $z = \tau_0 \in F$. Then

$$F \subset \mathrm{Aut}(\mathfrak{su}(n))^z = (\mathrm{O}(n)/\langle -I\rangle) \times \langle z\rangle.$$

By Lemma 2.18, we get that $m_z = m$. Then (ϵ_F, δ_F) coincides with $(\epsilon_{F'}, \epsilon_{F'})$ when $F' = F \cap \mathrm{Int}(\mathfrak{su}(n))$ is considered as a subgroup of $\mathrm{O}(n)/\langle -I\rangle$. Then the conclusion follows from Proposition 2.12 and Lemma 2.11.

Otherwise, for any $z \in F - F \cap \mathrm{Int}(\mathfrak{su}(n))$, we have that $z \sim \tau_0' = \tau_0 \mathrm{Ad}(J_{n/2})$. We may and do assumen that $z = \tau_0' \in F$. Then

$$F \subset \mathrm{Aut}(\mathfrak{su}(n))^z = (\mathrm{Sp}(n/2)/\langle -I \rangle) \times \langle z \rangle.$$

And we have $\mu_z \equiv 1$ since all elements in $F - F \cap \mathrm{Int}(\mathfrak{su}(n))$ are conjugate to τ_0'. Then the conjugacy class of F is determined by rank$F$ by Proposition 2.16. Moreover, in this case, we have $(\epsilon_F, \delta_F) = (0, 1)$ and rankA_F = rank$F - 1$. Then the tuple of invariants $(\epsilon_F, \delta_F, \mathrm{rank} F, \mathrm{rank} A_F)$ is different from that for any subgroup considered in the first case. The reason is: if a subgroup F in the first case satisfies rankA_F = rank$F - 1$, then its elements in $F - F \cap \mathrm{Int}(\mathfrak{su}(n))$ are all conjugate to τ_0, by which we have $(\epsilon_F, \delta_F) = (0, 0)$. □

2.4 A class of elementary abelian 2-subgroups and symplectic metric spaces

The elementary abelian 2-subgroups F of $\mathrm{O}(n)/\langle -I \rangle$ (or $\mathrm{Sp}(n)/\langle -I \rangle$) with non-identity elements all conjugate to $[I_{\frac{n}{2},\frac{n}{2}}]$, $[J_{\frac{n}{2}}]$ (or $[I_{\frac{n}{2},\frac{n}{2}}]$, $[\mathbf{i}I]$) have a particular nice shape.

Proposition 2.22 *For an elementary abelian 2-subgroup F of $\mathrm{O}(n)/\langle -I \rangle$ (or $\mathrm{Sp}(n)/\langle -I \rangle$), any non-identity element of F is conjugate to $[I_{\frac{n}{2},\frac{n}{2}}]$, $[J_{\frac{n}{2}}]$ (or $[I_{\frac{n}{2},\frac{n}{2}}]$, $[\mathbf{i}I]$) if and only if any non-identity element of A_F is conjugate to $[I_{\frac{n}{2},\frac{n}{2}}]$.*

Proof Since elements in $F - A_F$ are all conjugate to $[I_{\frac{n}{2},\frac{n}{2}}]$, $[J_{\frac{n}{2}}]$ (or $[I_{\frac{n}{2},\frac{n}{2}}]$, $[\mathbf{i}I]$) and any element of A_F is not conjugate to $[J_{\frac{n}{2}}]$ (or $[\mathbf{i}I]$), the conclusion follows. □

Regard A_F as a subgroup of $G' = \mathrm{O}(n')/\langle -I \rangle$, $\mathrm{U}(n')/\langle -I \rangle$ or $\mathrm{Sp}(n')/\langle -I \rangle$, where $n' = \frac{n}{2^{s+k}}$ $(k = 2, 1, 0)$. Then the condition of any non-identity element of A_F is conjugate to $[I_{\frac{n}{2},\frac{n}{2}}]$ in G is equivalent to any non-identity element of A_F is conjugate to $[I_{\frac{n'}{2},\frac{n'}{2}}]$ in G'.

Let $F^* = \mathrm{Hom}(F, \mathbb{F}_2)$ be the dual group of an elementary abelian 2-group.

For $n = 2^m s$ with s odd, let

$$K = \{\pm 1\}^n / \langle \underbrace{(-1, \ldots, -1)} \rangle.$$

This is an elementary abelian 2-group of rank $n - 1$. We want to characterize subgroups F of K such that any non-identity element $x \in F$ is of the form $x = [(x_1, x_2, \ldots, x_n)]$ with $x_i = -1$ for $\frac{n}{2}$ indices i and $x_i = 1$ for the other $\frac{n}{2}$ indices i.

Lemma 2.23 *For a subgroup F of K as above, let r be the rank of F as an elementary abelian 2-group. Then we can divide $J = \{1, 2, \ldots, n\}$ into a disjoint union of 2^r subsets*

$$\{J_\alpha : \alpha \in F^*\}$$

with each J_α of cardinality $\frac{n}{2^r} = 2^{m-r}s$ such that any element $x \in F$ is of the form

$$x = \underbrace{[(t_1, t_2, \ldots, t_n)]}, \ t_i = \alpha(x), \quad \forall i \in J_\alpha.$$

Proof Choose a subgroup F' of $\{\pm 1\}^n$ such that its projection to K has image equal to F and the projection map onto F is an isomorphism. Then any non-identity element $x \in F'$ is of the form $x = (x_1, x_2, \ldots, x_n)$ with $x_i = -1$ for $\frac{n}{2}$ indices i and $x_i = 1$ for the other $\frac{n}{2}$ indices i. For any $x \in F'$, let $x = (t_{x,1}, t_{x,2}, \ldots, t_{x,n})$, $t_{x,i} = \pm 1$. For an index i, the map $x \mapsto t_{x,i}$ is an homomorphism from F' to ± 1, so there exists $\alpha_i \in F'^*$ such that

$$t_{x,i} = \alpha_i(x), \forall x \in F'.$$

For any $\alpha \in F^*$, define

$$J_\alpha = \{1 \le i \le n | \alpha_i = \alpha\}.$$

Then $J = \{1, 2, \dots, n\}$ is a disjoint union of 2^r subsets $\{J_\alpha : \alpha \in F^*\}$ and any element $x \in F'$ is of the form $x = (t_1, t_2, \dots, t_n)$, $t_i = \alpha(x)$, $\forall i \in J_\alpha$. We show that the cardinarity of each J_α is $\frac{n}{2^r} = 2^{m-r}s$.

Let $\alpha_0 = 0 \in F'^*$ be the zero element. For any $\alpha \neq \alpha_0$, count the number of pairs (x, i) with $\alpha(x) = -1$ and $t_{x,i} = -1$. For a fixed $x \in F$, when $x \notin \ker \alpha$, there are $\frac{n}{2}$ such (x, i); when $x \in \ker \alpha$, there are no such (x, i). For a fixed i, $1 \le i \le n$, when $x \notin J_{\alpha_0} \cup J_\alpha$, i.e., $\alpha_i \neq \alpha_0$ and $\alpha_i \neq \alpha$, there are 2^{r-2} such (x, i); when $i \in J_\alpha$, i.e., $\alpha_i = \alpha$, there are 2^{r-1} such (x, i); when $i \in J_{\alpha_0}$, i.e., $\alpha_i = \alpha_0$, there are no such (x, i). Count the number of pairs (x, i) with $\alpha(x) = -1$ and $t_{x,i} = -1$ in two ways, we get an equality

$$2^{r-1}\frac{n}{2} = (n - |J_\alpha| - |J_{\alpha_0}|)2^{r-2} + |J_\alpha|2^{r-1}.$$

This implies that $|J_\alpha| = |J_{\alpha_0}|$. Then the cardinarity of each J_α is $\frac{n}{2^r} = 2^{m-r}s$.

Since the projection map from F' to F is an isomorphism, we can identify F'^* and F^*. Then we get the conclusion of the lemma. □

Proposition 2.24 *For an elementary abelian 2-subgroup F of* $\mathrm{O}(n)/\langle -I\rangle$ *(or $F \subset$* $\mathrm{Sp}(n)/\langle -I\rangle$*) with non-identity elements all conjugate to $[I_{\frac{n}{2},\frac{n}{2}}]$, $[J_{\frac{n}{2}}]$ (or $[I_{\frac{n}{2},\frac{n}{2}}]$, $[\mathbf{i}I]$), the conjugacy class of F is determined by the tuple $(\epsilon_F, \delta_F, r_F, s_F)$.*

Proof This follows from Propositions 2.12, 2.16, 2.22 and Lemma 2.23. □

Let $F_{r,s,\epsilon,\delta}$ be an elementary abelian 2-subgroup of $\mathrm{O}(n)/\langle -I\rangle$ (or $\mathrm{Sp}(n)/\langle -I\rangle$) satisfying the properties in Proposition 2.24 and with invariants (ϵ, δ, r, s), which is unique up to conjugation.

Definition 2.25 A finite-dimensional vector space V over the field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ is called a symplectic vector space if it is associated with a map $m : V \times V \longrightarrow \mathbb{F}_2$ such that $m(x, x) = 0$, $m(x, y) = m(y, x)$ and $m(x + y, z) = m(x, z)m(y, z)$ for any $x, y, x \in F$.

Moreover, it is called a symplectic metric space if there is another map $\mu : V \longrightarrow \mathbb{F}_2$ such that $\mu(0) = 0$ and $m(x, y) = \mu(x) + \mu(y) + \mu(x + y)$ for any $x, y \in V$.

Two symplectic vector spaces (V, m) and (V', m') are called isomorphic if there exists a linear space isomorphism $f : V \longrightarrow V'$ transferring m to m'.

Two symplectic metric spaces (V, m, μ) and (V', m', μ') are called isomorphic if there exists a linear space isomorphism $f : V \longrightarrow V'$ transferring (m, μ) to (m', μ').

The following proposition is clear.

Proposition 2.26 *The isomorphism class of a symplectic vector space* (V, m) *is determined by the dimensions* $(\dim_{\mathbb{F}_2} V, \dim_{\mathbb{F}_2} \ker m)$.

Definition 2.27 For a symplectic metric space V, define $A_V = \ker \mu|_{\ker m}$ and the defect index $\mathrm{defe}V = |\{x \in V : \mu(x) = 1\}| - |\{x \in V : \mu(x) = -1\}|$.

Define (ϵ_V, δ_V) as follows,

- When $\mu|_{\ker m} \neq 1$, define $(\epsilon_V, \delta_V) = (1, 0)$;
- when $\mu|_{\ker m} = 1$ and $\mathrm{defe}V < 0$, define $(\epsilon_V, \delta_V) = (0, 1)$;
- when $\mu|_{\ker m} = 1$ and $\mathrm{defe}V > 0$, define $(\epsilon_V, \delta_V) = (0, 0)$.

Define $r_V = \dim_{\mathbb{F}_2} A_V$, $s_V = \frac{1}{2}\dim_{\mathbb{F}_2}(V/\ker m) - \delta_V$.

Remark 2.28 When m is non-degenerate, μ is a non-degenerate quadratic form, in this case δ_V is the Arf invariant of μ.

The following proposition is an analogue of Proposition 2.24. And it also can be proved by the same method.

Proposition 2.29 *The isomorphism class of a symplectic metric space is determined by the invariants* $(r_V, s_V, \epsilon_V, \delta_V)$.
We have $\operatorname{defe} V = (1-\epsilon)(-1)^{\delta}2^{r+s+\delta}$.

Proposition 2.30 *For a vector space V over $\mathbb{F}_2$ of rank 3 with a map $\mu : V \longrightarrow \mathbb{F}_2$ satisfying $\mu(0) = 0$, let $m(x, y) = \mu(x) + \mu(y) + \mu(x + y)$. Then (V, m, μ) is a symplectic metric space if and only if m is bilinear, if and only if there are even number of elements in V with non-trivial values of the function μ.*

Proof With the definiton of m and the property $\mu(0) = 0$, we get the compatability relation and the property $m(x, x) = 0$, then (V, m, μ) is a symplectic metric if and only of m is a bilinear form. This is the first statement.

For any $x, y, z \in V$, when x, y, z are linearly dependent, the equality $m(x + y, z) = m(x, z) + m(y, z)$ follows from the definiton of m and the property $\mu(0) = 0$. When x, y, z are linearly independent, they consist in a basis of V. By the definition of m and the property $\mu(0) = 0$, we have that the equality $m(x + y, z) = m(x, z) + m(y, z)$ holds if and only if the sum of the values of μ over all elements of V is 0. That is also equivalent to there are even elements in V with μ-value 1. So the second statement follows. □

Let $V_{r,s;\epsilon,\delta}$ be a symplectic metric space with the prescribed invariants, which is unique up to isomorphism. Let $\mathrm{Sp}(r, s; \epsilon, \delta)$ be the group of automorphisms of $V_{r,s,\epsilon,\delta}$ preserving m and μ. Let $V_{s;\epsilon,\delta} = V_{0,s;\epsilon,\delta}$ and $\mathrm{Sp}(s; \epsilon, \delta) = Sp(0, s; \epsilon, \delta)$. It is clear that

$$\mathrm{Sp}(r, s; \epsilon, \delta) = \mathrm{Hom}(V_{s;\epsilon,\delta}, \mathbb{F}_2^r) \rtimes (\mathrm{Sp}(s; \epsilon, \delta) \times \mathrm{GL}(\mathbb{F}_2^r)).$$

Let $\mathrm{Sp}(s) = \mathrm{Sp}(s, \mathbb{F}_2)$ be the degree-s symplectic group over the field $\mathbb{F}_2$.

Proposition 2.31 *We have the following formulas for the orders of* $\mathrm{Sp}(s; \epsilon, \delta)$,

$$|\mathrm{Sp}(s; 0, 0)| = \left(\prod_{1\le i\le s-1} (2^{i+1} - 1)(2^i + 1)\right) \cdot 2^{s^2-s+1},$$

$$|\mathrm{Sp}(s-1; 0, 1)| = 3 \cdot \left(\prod_{1\le i\le s-1} (2^i - 1)(2^{i+1} + 1)\right) \cdot 2^{s^2-s+1},$$

$$|\mathrm{Sp}(s; 1, 0)| = |\mathrm{Sp}(s)| = \left(\prod_{1\le i\le s} (2^i - 1)(2^i + 1)\right) 2^{s^2}.$$

Proof When $s = 1$ or 0, these are clear. So we just need to calculate

$$|\mathrm{Sp}(s; \epsilon, \delta)|/|\mathrm{Sp}(s-1; \epsilon, \delta)|.$$

We calculate it for the case $\epsilon = \delta = 0$, the other cases are similar.

$\mathrm{Sp}(s; 0, 0)$ permutes the non-identity elements $x \in V_{s;0,0}$ with $\mu(x) = 0$, there are $\frac{2^{2s}+2^s}{2} - 1 = (2^s - 1)(2^{s-1} + 1)$ such elements. Fix two distinct non-identity elements $x_1, x_2 \in V_{s,0,0}$ with $\mu(x_1) = \mu(x_2) = 0$ and $m(x_1, x_2) = 1$. For any other x with $\mu(x) = 0$

and $(x_1,x)=1$, (x_1,x) is transformed to (x_1,x_2) under some transformation in $\mathrm{Sp}(s;0,0)$. Fixing x_1, there are 2^{2s-2} such elements x. Moreover, the subgroup of $\mathrm{Sp}(s;0,0)$ consisting of elements fixing x_1 and x_2 is isomorphic to $\mathrm{Sp}(s-1;0,0)$. So So we have $|\mathrm{Sp}(s;\epsilon,\delta)|/|\mathrm{Sp}(s-1;\epsilon,\delta)|=(2^s-1)(2^{s-1}+1)2^{2s-2}$. □

Since we have

$$V_{s;0,0}\oplus V_{0;1,0}\cong V_{s-1;0,1}\oplus V_{0;1,0}\cong V_{s;1,0},$$

so we can regard $\mathrm{Sp}(s;0,0)$ and $\mathrm{Sp}(s-1;0,1)$ as subgroups of $\mathrm{Sp}(s;1,0)$.

Proposition 2.32 $\mathrm{Sp}(s;1,0)\cong\mathrm{Sp}(s)$.

Proof Since $V_{s;1,0}/\ker m=\mathbb{F}_2^{2s}$ is a symplectic vector space of dimension $2s$, by restriction we get a natural homomorphism $p:\mathrm{Sp}(s;1,0)\longrightarrow\mathrm{Sp}(s)$.

Let z be the unique non-identity element in $\ker m$. Suppose that $p(f)=1$ for some $f\in\mathrm{Sp}(s;1,0)$, then for any $x\in V_{s;1,0}$, $f(x)=x$ or $f(x)=xz$. Since $\mu(xz)=\mu(x)+\mu(z)+m(x,z)=\mu(x)+1$, so $f(x)\neq xz$. Then $f(x)=x$ for any $x\in V_{s;1,0}$. Thus p is injective.

Moreover, by Proposition 2.31 we have $|\mathrm{Sp}(s;1,0)|=|\mathrm{Sp}(s)|$. So p is an isomorphism. □

Since an element in $\mathrm{Sp}(s;0,0)$ or $\mathrm{Sp}(s-1;0,1)$ preserves the symplectic form m on $V=\mathbb{F}_2^{2s}$, so we have inclusions $\mathrm{Sp}(s;0,0)\subset\mathrm{Sp}(s)$ and $\mathrm{Sp}(s-1;0,1)\subset\mathrm{Sp}(s)$.

Proposition 2.33 *We have*

$$[\mathrm{Sp}(s):\mathrm{Sp}(s;0,0)]=2^{s-1}(2^s+1)$$

and

$$[\mathrm{Sp}(s):\mathrm{Sp}(s-1;0,1)]=2^{s-1}(2^s-1).$$

Proof This follows from Proposition 2.31 directly. □

Define the groups $\mathrm{Sp}(s;t)$ $(s,t\geq 0)$ as the automorphism group a symplecic vector space (V,m) over $\mathbb{F}_2$ with $\operatorname{rank} V=2s+t$ and $\operatorname{rank}\ker m=t$. It is clear that $\mathrm{Sp}(s;0)=\mathrm{Sp}(s)$ and

$$\mathrm{Sp}(s;t)=\mathrm{Hom}(\mathbb{F}_2^{2s},\mathbb{F}_2^t)\rtimes(\mathrm{GL}(t,\mathbb{F}_2)\times\mathrm{Sp}(s)).$$

3 Exceptional compact simple Lie groups (algebras)

3.1 Complex semi-simple Lie algebra and a specific compact real form

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{g}$ has a root-space decomposition

$$\mathfrak{g}=\mathfrak{h}\oplus\left(\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha\right),$$

where $\Delta=\Delta(\mathfrak{g},\mathfrak{h})$ is the root system of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is the root space of a root $\alpha\in\Delta$. Let B be the Killing form on $\mathfrak{g}$. It is a non-degenerate symmetric form. The restriction of B to

$\mathfrak{h}$ is also non-degenerate. Let $\mathfrak{h}^*$ be the dual complex vector space of $\mathfrak{h}$. For any $\lambda \in \mathfrak{h}^*$, let $H_\lambda \in \mathfrak{h}$ be the element in $\mathfrak{h}$ determined uniquely by

$$B(H_\lambda, H) = \lambda(H), \quad \forall H \in \mathfrak{h}.$$

For any $\lambda, \mu \in \mathfrak{h}^*$, let $\langle \lambda, \mu \rangle := B(H_\lambda, H_\mu)$. Then $\langle \cdot, \cdot \rangle$ is an inner product on $\mathfrak{h}^*$.

For any root α, we have

$$H_\alpha \in \mathfrak{h}. \tag{1}$$

Define

$$H'_\alpha = \frac{2}{\alpha(H_\alpha)} H_\alpha, \tag{2}$$

which is called a co-root; let

$$0 \neq X_\alpha \in \mathfrak{g}_\alpha \tag{3}$$

be any non-zero vector (recall that $\dim \mathfrak{g}_\alpha = 1$), which is called a root vector of the root α. The notations $H_\alpha,\ H'_\alpha,\ X_\alpha$ will be used frequently in this paper.

Note that, for any $\alpha, \beta \in \Delta$,

$$\begin{aligned}\langle \alpha, \beta \rangle &= B(H_\alpha, H_\beta) = \beta(H_\alpha) = \alpha(H_\beta) \in \mathbb{R},\\ \langle \alpha, \alpha \rangle &= B(H_\alpha, H_\alpha) = \alpha(H_\alpha) \neq 0,\end{aligned}$$

and $2\langle \alpha, \beta \rangle / \langle \beta, \beta \rangle \in \mathbb{Z}$. We also note that

$$\mathrm{span}_{\mathbb{R}}\{\alpha | \alpha \in \Delta\} \subset \mathfrak{h}^*$$

is a real vector space of dimension equal to $r = \mathrm{rank}\mathfrak{g} = \dim_{\mathbb{C}} \mathfrak{h}$ (cf. [9, Pages 140–162]).

We set

$$A_{\alpha,\beta} = 2\langle \alpha, \beta \rangle / \langle \beta, \beta \rangle = \alpha(H'_\beta).$$

Then

$$[H'_\alpha, X_\beta] = \beta(H'_\alpha) X_\beta = \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} X_\beta = A_{\beta,\alpha} X_\beta.$$

Choose a lexicography order of $\mathrm{span}_{\mathbb{R}}\{\alpha | \alpha \in \Delta\}$ to get a positive system Δ^+ and a simple system Π. Let

$$\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}. \tag{4}$$

For brevity, we write

$$H_i,\ H'_i \tag{5}$$

instead of $H_{\alpha_i},\ H'_{\alpha_i}$ for a simple root α_i.

Draw $A_{\alpha,\beta} A_{\beta,\alpha}$ edges to connect any two distinct simple roots α and β, and draw an arrow from α to β if $\langle \alpha, \alpha \rangle > \langle \beta, \beta \rangle$, we get a graph. This graph is connected if and only if $\mathfrak{g}$ is a simple Lie algebra, in this case it is called the Dynkin diagram of $\mathfrak{g}$. We always follow the Bourbaki numbering to order the simple roots (cf. [8, Page 3]).

Let $\mathrm{Aut}(\mathfrak{g})$ be the group of all complex linear automorphisms of $\mathfrak{g}$ and $\mathrm{Int}(\mathfrak{g})$ be the subgroup of inner automorphisms. We define

$$\mathrm{Out}(\mathfrak{g}) := \mathrm{Aut}(\mathfrak{g}) / \mathrm{Int}(\mathfrak{g}).$$

The exponential map $\exp : \mathfrak{g} \longrightarrow \mathrm{Aut}(\mathfrak{g})$ is given by

$$\exp(X) = \exp(\mathrm{ad}(X)), \quad \forall X \in \mathfrak{g} = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})),$$

where $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ is defined by $\mathrm{ad}(X)(Y) = [X, Y]$, $\forall Y \in \mathfrak{g}$.

One can normalize the root vectors $\{X_\alpha, X_{-\alpha}\}$ so that $B(X_\alpha, X_{-\alpha}) = 2/\alpha(H_\alpha)$. Then $[X_\alpha, X_{-\alpha}] = H'_\alpha$. Moreover, one can normalize $\{X_\alpha\}$ appropriately, such that

$$\mathfrak{u}_0 = \mathrm{span}_{\mathbb{R}}\{X_\alpha - X_{-\alpha}, i(X_\alpha + X_{-\alpha}), iH_\alpha : \alpha \in \Delta^+\} \tag{6}$$

is a compact real form of $\mathfrak{g}$ ([9, Pages 348–354]). Define

$$\theta(X + iY) := X - iY, \quad \forall X, Y \in \mathfrak{u}_0.$$

Then θ is a Cartan involution of $\mathfrak{g}$ (as a real semisimple Lie algebra) and $\mathfrak{u}_0 = \mathfrak{g}^\theta$ is a maximal compact subalgebra of $\mathfrak{g}$. Any other compact real form of $\mathfrak{g}$ is conjugate to $\mathfrak{u}_0$. *In the below, whenever we discuss a compact real form of* $\mathfrak{g}$, *we always use this compact real form* $\mathfrak{u}_0$ *in* (6).

Let $\mathrm{Aut}(\mathfrak{u}_0)$ be the group of automorphisms of $\mathfrak{u}_0$ and $\mathrm{Int}(\mathfrak{u}_0) = \mathrm{Aut}(\mathfrak{u}_0)_0$ be the subgroup of inner automorphisms. Any automorphism of $\mathfrak{u}_0$ extends uniquely to a holomorphic automorphism of $\mathfrak{g}$, so $\mathrm{Aut}(\mathfrak{u}_0) \subset \mathrm{Aut}(\mathfrak{g})$. Similarly we have $\mathrm{Int}(\mathfrak{u}_0) \subset \mathrm{Int}(\mathfrak{g})$. Define

$$\Theta(f) := \theta f \theta^{-1}, \quad \forall f \in \mathrm{Aut}(\mathfrak{g}).$$

Then it is a Cartan involution of $\mathrm{Aut}(\mathfrak{g})$ with differential θ. It follows that $\mathrm{Aut}(\mathfrak{u}_0) = \mathrm{Aut}(\mathfrak{g})^\Theta$ and $\mathrm{Int}(\mathfrak{u}_0) = \mathrm{Int}(\mathfrak{g})^\Theta$ are maximal compact subgroups of $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Int}(\mathfrak{g})$ respectively. We also have

$$\mathrm{Out}(\mathfrak{u}_0) := \mathrm{Aut}(\mathfrak{u}_0)/\mathrm{Int}(\mathfrak{u}_0) \cong \mathrm{Out}(\mathfrak{g}) \cong \mathrm{Aut}(\Pi),$$

where $\mathrm{Aut}(\Pi)$ is the symmetry group of the graph Π consisting of permutations of vertices preserving the multiples of edges and directions of arrows.

3.2 Involutions

Let $\mathfrak{u}_0$ be a compact simple Lie algebra and $G = \mathrm{Aut}(\mathfrak{u}_0)$ be its automorphism group. The conjugacy classes of involutions in G are in one-one correspondence with the isomorphism classes of real forms of the complexified Lie algebra $\mathfrak{g} = \mathfrak{u}_0 \otimes_{\mathbb{R}} \mathbb{C}$, and also in one-one correspondence with compact irreducible Riemannian symmetric pairs $(\mathfrak{u}_0, \mathfrak{h}_0)$. These objects were classified by Élie Cartan in 1920s. We give representatives of conjugacy classes of involutions in the automorphism group $G = \mathrm{Aut}(\mathfrak{u}_0)$ for each compact simple exceptional Lie algebra $\mathfrak{u}_0$. The following are from [8, Pages 5–6]. In particular, as explained in [8], the notation $\mathfrak{e}_{6,-2}$ denotes a real simple Lie algebra with a Cartan decomposition $\mathfrak{u}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ such that $\mathfrak{g} = \mathfrak{u}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is a complex simple Lie algebra of type $\mathbf{E_6}$ and $\dim \mathfrak{k}_0 - \dim \mathfrak{p}_0 = -2$, and similarly for the notations of other real simple Lie algebras.

(i) Type $\mathbf{E_6}$. For $\mathfrak{u}_0 = \mathfrak{e}_6$, let τ be a specific diagram involution defined by

$$\begin{aligned}
&\tau(H_{\alpha_1}) = H_{\alpha_6}, \quad \tau(H_{\alpha_6}) = H_{\alpha_1}, \quad \tau(H_{\alpha_3}) = H_{\alpha_5}, \quad \tau(H_{\alpha_5}) = H_{\alpha_3},\\
&\tau(H_{\alpha_2}) = H_{\alpha_2}, \quad \tau(H_{\alpha_4}) = H_{\alpha_4}, \quad \tau(X_{\pm\alpha_1}) = X_{\pm\alpha_6}, \quad \tau(X_{\pm\alpha_6}) = X_{\pm\alpha_1},\\
&\tau(X_{\pm\alpha_3}) = X_{\pm\alpha_5}, \quad \tau(X_{\pm\alpha_5}) = X_{\pm\alpha_3}, \quad \tau(X_{\pm\alpha_2}) = X_{\pm\alpha_2}, \quad \tau(X_{\pm\alpha_4}) = X_{\pm\alpha_4}.
\end{aligned}$$

Let

$$\sigma_1 = \exp(\pi i H'_2), \sigma_2 = \exp(\pi i(H'_1 + H'_6)), \sigma_3 = \tau, \sigma_4 = \tau \exp(\pi i H'_2).$$

Then $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$, which correspond to Riemannian symmetric spaces of type **EII, EIII, EIV, EI** and the corresponding real forms are $\mathfrak{e}_{6,-2}, \mathfrak{e}_{6,14}, \mathfrak{e}_{6,26}, \mathfrak{e}_{6,-6}$. σ_1, σ_2 are inner automorphisms, σ_3, σ_4 are outer automorphisms.

(ii) Type $\mathbf{E_7}$. For $\mathfrak{u}_0 = \mathfrak{e}_7$, let

$$\sigma_1 = \exp\left(\pi i H_2'\right), \quad \sigma_2 = \exp\left(\pi i \frac{H_2' + H_5' + H_7'}{2}\right),$$
$$\sigma_3 = \exp\left(\pi i \frac{H_2' + H_5' + H_7' + 2H_1'}{2}\right).$$

Then $\sigma_1, \sigma_2, \sigma_3$ represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$, which correspond to Riemannian symmetric spaces of type **EVI, EVII, EV** and the corresponding real forms are $\mathfrak{e}_{7,3}, \mathfrak{e}_{7,25}, \mathfrak{e}_{7,-7}$.

(iii) Type $\mathbf{E_8}$. For $\mathfrak{u}_0 = \mathfrak{e}_8$, let

$$\sigma_1 = \exp(\pi i H_2'), \sigma_2 = \exp(\pi i (H_2' + H_1')).$$

Then σ_1, σ_2 represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$, which correspond to Riemannian symmetric spaces of type **EIX, EVIII** and the corresponding real forms are $\mathfrak{e}_{8,24}, \mathfrak{e}_{8,-8}$.

(iv) Type $\mathbf{F_4}$. For $\mathfrak{u}_0 = \mathfrak{f}_4$, let

$$\sigma_1 = \exp(\pi i H_1'), \sigma_2 = \exp(\pi i H_4').$$

Then σ_1, σ_2 represent all conjugacy classes of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$, which correspond to Riemannian symmetric spaces of type **FI, FII** and the corresponding real forms are $\mathfrak{f}_{4,-4}, \mathfrak{f}_{4,20}$.

(v) Type $\mathbf{G_2}$. For $\mathfrak{u}_0 = \mathfrak{g}_2$, let $\sigma = \exp(\pi H_1')$, which represents the unique conjugacy class of involutions in $\mathrm{Aut}(\mathfrak{u}_0)$, corresponds to Riemannian symmetric space of type **G** and the corresponding real form is $\mathfrak{g}_{2,-2}$.

We remark that, in types $\mathbf{E_8}$, $\mathbf{F_4}$, $\mathbf{G_2}$, the automorphism groups of the simple Lie algebras, $\mathrm{Aut}(\mathfrak{e}_8)$, $\mathrm{Aut}(\mathfrak{f}_4)$, $\mathrm{Aut}(\mathfrak{g}_2)$, are connected and simply connected. In type $\mathbf{E_6}$, $\mathrm{Aut}(\mathfrak{e}_6)$ is not connected and $\mathrm{Int}(\mathfrak{e}_6)$ is not simply connected, the image of the adjoint homomorphism $\pi : \mathrm{E}_6 \longrightarrow \mathrm{Aut}(\mathfrak{e}_6)$ is $\mathrm{Int}(\mathfrak{e}_6)$ and the kernel of π (i.e., $Z(\mathrm{E}_6)$) is of order 3. Since $\mathrm{Int}(\mathfrak{e}_6)$ has two conjugacy classes of involutions, so E_6 has two conjugacy classes of involutions. Their representatives $\sigma_1' = \exp(\pi i H_2')$, $\sigma_2' = \exp(\pi i (H_1' + H_6'))$. Here $\exp : \mathfrak{e}_6 \longrightarrow \mathrm{E}_6$ is the exponential map for the Lie group E_6. In type $\mathbf{E_7}$, $\mathrm{Aut}(\mathfrak{e}_7)$ is connected but not simply connected, the adjoint homomorphism $\pi : \mathrm{E}_7 \longrightarrow \mathrm{Aut}(\mathfrak{e}_7)$ is surjective and the kernel of π (i.e., $Z(\mathrm{E}_7)$) is of order 2. The preimages of $\sigma_2, \sigma_3 \in \mathrm{Aut}(\mathfrak{e}_7)$ in E_7 are elements of order 4; and the preimages of σ_1 are two non-conjugate involutions. So E_7 has two conjugacy classes of involutions. Their representatives are $\sigma_1' = \exp(\pi i H_2')$ and $\sigma_2' = \exp(\pi i (H_1' + H_6'))$. Here $\exp : \mathfrak{e}_7 \longrightarrow \mathrm{E}_7$ is the exponential map for the Lie group E_7.

There is an ascending sequence

$$\mathrm{F}_4 \subset \mathrm{E}_6 \subset \mathrm{E}_7 \subset \mathrm{E}_8,$$

we observe that under these inclusions, the involutions σ_2 in F_4 (σ_2' in E_6, or σ_2' in E_7) is mapped to conjugate element of the involution σ_2' in E_6 (σ_2' in E_7, or σ_2 in E_8). The conjugacy class containing σ_2 (or σ_2') in each type is particularly important to us as we will use them to define the translation subgroup A_F for an elementary abelian 2-subgroup F.

The following Table 2 is from Tables 1 and 2 in [8], which describes the isomorphism type of the symmetric subgroup $\mathrm{Aut}(\mathfrak{u}_0)^{\theta}$ and the isotropic module $\mathfrak{p} = \mathfrak{g}^{-\theta}$ for each pair $(\mathfrak{u}_0, \theta)$ with $\mathfrak{u}_0$ a compact exceptional simple Lie algebra and θ an involtution in $\mathrm{Aut}(\mathfrak{u}_0)$.

Remark 3.1 We apologize that we use σ_i to represent the conjugacy classes of involutions in the automorphism groups $\mathrm{Aut}(\mathfrak{u}_0)$ in all types (as well as use σ_i' to represent the conjugacy classes of involutions in the connected and simply connected compact Lie groups E_6 and E_7). But this causes no ambiguity as we always specify in which group we are talking about conjugacy classes.

3.3 Klein four subgroups

In [8, Section 4], we constructed some Klein four subgroups of $\mathrm{Aut}(\mathfrak{u}_0)$ and described the conjugacy classes of involutions in them, it is showed that they represent all conjugacy classes of Klein four subgroups. These Klein fours subgroups, as well as their fixed point subalgebras and their involutions types (cf. Definition 3.3) are listed in Table 3.

From Table 3, we see that, the groups $\mathrm{Aut}(\mathfrak{e}_6)$, $\mathrm{Aut}(\mathfrak{e}_7)$, $\mathrm{Aut}(\mathfrak{e}_8)$, $\mathrm{Aut}(\mathfrak{f}_4)$, $\mathrm{Aut}(\mathfrak{g}_2)$ have 8, 8, 4, 3, 1 conjugacy classes of Klein four subgroups in them respectively. Most of these conjugacy classes are distinguished by their involution types (Definition 3.3) except that the Klein four subgroups Γ_1, Γ_2 of $\mathrm{Aut}(\mathfrak{e}_7)$ have the same involution type [both are $(\sigma_1, \sigma_1, \sigma_1)$]. The Klein four subgroups $\Gamma_1, \Gamma_2 \subset \mathrm{Aut}(\mathfrak{e}_7)$ can be characterized in this way: a Klein four subgroup $F \subset \mathrm{E}_7$ with $\pi(F) = \Gamma_1$ [or $\pi(F) = \Gamma_2$] have an odd number of elements (or an even number of elements) conjugate to σ_2', where $\pi : \mathrm{E}_7 \longrightarrow \mathrm{Aut}(\mathfrak{e}_7)$ is the adjoint homomorphism, which is a double covering. That is equivalent to say, we can choose a Klein

Table 2 Symmetric subgroups and isotropic modules

	$\mathfrak{u}_0$	θ	$\mathrm{Aut}(\mathfrak{u}_0)^{\theta}$	$\mathfrak{p}$
EI	$\mathfrak{e}_6$	$\sigma_4 = \tau\exp(\pi i H_2')$	$(\mathrm{Sp}(4)/\langle -1\rangle) \times \langle\theta\rangle$	V_{ω_4}
EII	$\mathfrak{e}_6$	$\sigma_1 = \exp(\pi i H_2')$	$(\mathrm{SU}(6)\times\mathrm{Sp}(1)/\langle(e^{\frac{2\pi i}{3}}I, 1), (-I,-1)\rangle) \rtimes \langle\tau\rangle$	$\wedge^3\mathbb{C}^6 \otimes \mathbb{C}^2$
			$\tau^2 = 1,\ \mathfrak{k}_0^{\tau} = \mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$	
EIII	$\mathfrak{e}_6$	$\sigma_2 = \exp(\pi i(H_1' + H_6'))$	$(\mathrm{Spin}(10)\times\mathrm{U}(1)/\langle(c, i)\rangle) \rtimes \langle\tau\rangle$	$(M_+ \otimes 1) \oplus (M_- \otimes \bar{1})$
			$\tau^2 = 1,\ \mathfrak{k}_0^{\tau} = \mathfrak{so}(9)$	
EIV	$\mathfrak{e}_6$	$\sigma_3 = \tau$	$\mathrm{F}_4 \times \langle\theta\rangle$	V_{ω_4}
EV	$\mathfrak{e}_7$	$\sigma_3 = \exp(\pi i(H_1' + H_0'))$	$(\mathrm{SU}(8)/\langle iI\rangle) \rtimes \langle\omega\rangle$	$\wedge^4\mathbb{C}^8$
			$\omega^2 = 1,\ \mathfrak{k}_0^{\omega} = \mathfrak{sp}(4)$	
EVI	$\mathfrak{e}_7$	$\sigma_1 = \exp(\pi i H_2')$	$(\mathrm{Spin}(12)\times\mathrm{Sp}(1))/\langle(c, 1), (-1, -1)\rangle$	$M_+ \otimes \mathbb{C}^2$
EVII	$\mathfrak{e}_7$	$\sigma_2 = \exp(\pi i H_0')$	$((\mathrm{E}_6\times\mathrm{U}(1))/\langle(c, e^{\frac{2\pi i}{3}})\rangle) \rtimes \langle\omega\rangle$	$(V_{\omega_1} \otimes 1) \oplus (V_{\omega_6} \otimes \bar{1})$
			$\omega^2 = 1,\ \mathfrak{k}_0^{\omega} = \mathfrak{f}_4$	
EVIII	$\mathfrak{e}_8$	$\sigma_2 = \exp(\pi i(H_1' + H_2'))$	$\mathrm{Spin}(16)/\langle c\rangle$	M_+
EIX	$\mathfrak{e}_8$	$\sigma_1 = \exp(\pi i H_1')$	$\mathrm{E}_7\times\mathrm{Sp}(1)/\langle(c, -1)\rangle$	$V_{\omega_7} \otimes \mathbb{C}^2$
FI	$\mathfrak{f}_4$	$\sigma_1 = \exp(\pi i H_1')$	$(\mathrm{Sp}(3)\times\mathrm{Sp}(1))/\langle(-I, -1)\rangle$	$V_{\omega_3} \otimes \mathbb{C}^2$
FII	$\mathfrak{f}_4$	$\sigma_2 = \exp(\pi i H_4')$	$\mathrm{Spin}(9)$	M
G	$\mathfrak{g}_2$	$\sigma = \exp(\pi i H_1')$	$(\mathrm{Sp}(1)\times\mathrm{Sp}(1))/\langle(-1, -1)\rangle$	$\mathrm{Sym}^3\mathbb{C}^2 \otimes \mathbb{C}^2$

Table 3 Klein four subgroups in $\mathrm{Aut}(\mathfrak{u}_0)$ for exceptional case

$\mathfrak{u}_0$	Γ_i	$\mathfrak{k}_0 = \mathfrak{u}_0^{\Gamma_i}$	Involution type
$\mathfrak{e}_6$	$\Gamma_1 = \langle \exp(\pi i H_2'), \exp(\pi i H_4') \rangle$	$(\mathfrak{su}(3))^2 \oplus (i\mathbb{R})^2$	$(\sigma_1, \sigma_1, \sigma_1)$
$\mathfrak{e}_6$	$\Gamma_2 = \langle \exp(\pi i H_4'), \exp(\pi i (H_3' + H_4' + H_5')) \rangle$	$\mathfrak{su}(4) \oplus (\mathfrak{sp}(1))^2 \oplus i\mathbb{R}$	$(\sigma_1, \sigma_1, \sigma_2)$
$\mathfrak{e}_6$	$\Gamma_3 = \langle \exp(\pi i (H_2' + H_1')), \exp(\pi i (H_4' + H_1')) \rangle$	$\mathfrak{su}(5) \oplus (i\mathbb{R})^2$	$(\sigma_1, \sigma_2, \sigma_2)$
$\mathfrak{e}_6$	$\Gamma_4 = \langle \exp(\pi i (H_1' + H_6')), \exp(\pi i (H_3' + H_5')) \rangle$	$\mathfrak{so}(8) \oplus (i\mathbb{R})^2$	$(\sigma_2, \sigma_2, \sigma_2)$
$\mathfrak{e}_6$	$\Gamma_5 = \langle \exp(\pi i H_2'), \tau \rangle$	$\mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$	$(\sigma_1, \sigma_3, \sigma_4)$
$\mathfrak{e}_6$	$\Gamma_6 = \langle \exp(\pi i H_2'), \tau \exp(\pi i H_4') \rangle$	$\mathfrak{so}(6) \oplus i\mathbb{R}$	$(\sigma_1, \sigma_4, \sigma_4)$
$\mathfrak{e}_6$	$\Gamma_7 = \langle \exp(\pi i (H_1' + H_6'))), \tau \rangle$	$\mathfrak{so}(9)$	$(\sigma_2, \sigma_3, \sigma_3)$
$\mathfrak{e}_6$	$\Gamma_8 = \langle \exp(\pi i (H_1' + H_6')), \tau \exp(\pi i H_2') \rangle$	$\mathfrak{so}(5) \oplus \mathfrak{so}(5)$	$(\sigma_2, \sigma_4, \sigma_4)$
$\mathfrak{e}_7$	$\Gamma_1 = \langle \exp(\pi i H_2'), \exp(\pi i H_4') \rangle$	$\mathfrak{su}(6) \oplus (i\mathbb{R})^2$	$(\sigma_1, \sigma_1, \sigma_1)$
$\mathfrak{e}_7$	$\Gamma_2 = \langle \exp(\pi i H_2'), \exp(\pi i H_3') \rangle$	$\mathfrak{so}(8) \oplus (\mathfrak{sp}(1))^3$	$(\sigma_1, \sigma_1, \sigma_1)$
$\mathfrak{e}_7$	$\Gamma_3 = \langle \exp(\pi i H_2'), \tau \rangle$	$\mathfrak{so}(10) \oplus (i\mathbb{R})^2$	$(\sigma_1, \sigma_2, \sigma_2)$
$\mathfrak{e}_7$	$\Gamma_4 = \langle \exp(\pi i H_1'), \tau \rangle$	$\mathfrak{su}(6) \oplus \mathfrak{sp}(1) \oplus i\mathbb{R}$	$(\sigma_1, \sigma_2, \sigma_3)$
$\mathfrak{e}_7$	$\Gamma_5 = \langle \exp(\pi i H_2'), \tau \exp(\pi i H_1') \rangle$	$\mathfrak{su}(4) \oplus \mathfrak{su}(4) \oplus i\mathbb{R}$	$(\sigma_1, \sigma_3, \sigma_3)$
$\mathfrak{e}_7$	$\Gamma_6 = \langle \tau, \omega^{\mathrm{a}} \rangle$	$\mathfrak{f}_4$	$(\sigma_2, \sigma_2, \sigma_2)$
$\mathfrak{e}_7$	$\Gamma_7 = \langle \tau, \omega \exp(\pi i H_1') \rangle$	$\mathfrak{sp}(4)$	$(\sigma_2, \sigma_3, \sigma_3)$
$\mathfrak{e}_7$	$\Gamma_8 = \langle \tau \exp(\pi i H_1'), \omega \exp(\pi i H_3') \rangle$	$\mathfrak{so}(8)$	$(\sigma_3, \sigma_3, \sigma_3)$
$\mathfrak{e}_8$	$\Gamma_1 = \langle \exp(\pi i H_2'), \exp(\pi i H_4') \rangle$	$\mathfrak{e}_6 \oplus (i\mathbb{R})^2$	$(\sigma_1, \sigma_1, \sigma_1)$
$\mathfrak{e}_8$	$\Gamma_2 = \langle \exp(\pi i H_2'), \exp(\pi i H_1') \rangle$	$\mathfrak{so}(12) \oplus (\mathfrak{sp}(1))^2$	$(\sigma_1, \sigma_1, \sigma_2)$
$\mathfrak{e}_8$	$\Gamma_3 = \langle \exp(\pi i H_2'), \exp(\pi i (H_1' + H_4')) \rangle$	$\mathfrak{su}(8) \oplus i\mathbb{R}$	$(\sigma_1, \sigma_2, \sigma_2)$
$\mathfrak{e}_8$	$\Gamma_4 = \langle \exp(\pi i (H_2' + H_1')), \exp(\pi i (H_5' + H_1')) \rangle$	$\mathfrak{so}(8) \oplus \mathfrak{so}(8)$	$(\sigma_2, \sigma_2, \sigma_2)$
$\mathfrak{f}_4$	$\Gamma_1 = \langle \exp(\pi i H_2'), \exp(\pi i H_1') \rangle$	$\mathfrak{su}(3) \oplus (i\mathbb{R})^2$	$(\sigma_1, \sigma_1, \sigma_1)$
$\mathfrak{f}_4$	$\Gamma_2 = \langle \exp(\pi i H_3'), \exp(\pi i H_2') \rangle$	$\mathfrak{so}(5) \oplus (\mathfrak{sp}(1))^2$	$(\sigma_1, \sigma_1, \sigma_2)$
$\mathfrak{f}_4$	$\Gamma_3 = \langle \exp(\pi i H_4'), \exp(\pi i H_3') \rangle$	$\mathfrak{so}(8)$	$(\sigma_2, \sigma_2, \sigma_2)$
$\mathfrak{g}_2$	$\Gamma = \langle \exp(\pi i H_1'), \exp(\pi i H_2') \rangle$	$(i\mathbb{R})^2$	(σ, σ, σ)

[a] $\omega = \exp\left(\frac{\pi(X_{\alpha_2} - X_{-\alpha_2})}{2}\right) \exp\left(\frac{\pi(X_{\alpha_5} - X_{-\alpha_5})}{2}\right) \exp\left(\frac{\pi(X_{\alpha_7} - X_{-\alpha_7})}{2}\right)$

four subgroup $F \subset \mathrm{E}_7$ with $\pi(F) = \Gamma_1$ (or $\pi(F) = \Gamma_2$) such that all of its involutions are conjugate to σ_1' (or σ_2').

Given a Klein four subgroup $F \subset G$, we have six different pairs (θ, σ) generating F, but some of them may be conjugate.

Theorem 3.2 [8, Theorem 5.2] *Let* (θ, σ), (θ', σ') *be two pairs of commuting involutions in* $\mathrm{Aut}(\mathfrak{u}_0)$ *for* $\mathfrak{u}_0$ *a compact exceptional simple Lie algebra, then they are conjugate if and only if*

$$\theta \sim \theta', \sigma \sim \sigma', \theta\sigma \sim \theta'\sigma'$$

and the Klein four subgroups $\langle \theta, \sigma \rangle$, $\langle \theta', \sigma' \rangle$ *are conjugate.*

We remark that, $\mathrm{Aut}(\mathfrak{e}_7)$ has two non-conjugate Klein four subgroups with involutions all conjugate to σ_1, so the condition of "the Klein four subgroups $\langle \theta, \sigma \rangle$, $\langle \theta', \sigma' \rangle$ are conjugate" is necessary. By Theorem 3.2, Table 3 also classifies conjugacy classes of ordered pairs of commuting involutions in $\mathrm{Aut}(\mathfrak{u}_0)$. Which is another approach to Berger 's classification of semisimple symmetric pairs (cf. [3]).

3.4 An outline of the method of the classification

In type $\mathbf{G_2}$, it turns out the conjugacy class of an elementary abelian 2-subgroup of $\mathrm{Aut}(\mathfrak{g}_2)$ is determined by its rank and the rank is at most 3. So we have four conjugacy classes of elementary abelian 2-subgroups in total.

In type $\mathbf{F_4}$, by Table 3, we see that $\mathrm{Aut}(\mathfrak{f}_4)$ does not possess any Klein four subgroup with involution type $(\sigma_1, \sigma_2, \sigma_2)$. That implies, the subset consisting of the identity element and all elements conjugate to $\sigma_2$in an elementary abelian 2-subgroup F of $\mathrm{Aut}(\mathfrak{f}_4)$ is a subgroup of F. Let A_F be this subgroup. Then $r = \mathrm{rank} A_F$ and $s = \mathrm{rank} F/A_F$ are conjugate invariant. We show that the conjugacy class of F is determined by the pair (r, s) and the range of the pairs is $\{(r, s) : r \leq 2, s \leq 3\}$. So we have twelve conjugacy classes of elementary abelian 2-subgroups in total.

In type $\mathbf{E_6}$, we divide the elementary abelian 2-subgroups F of $\mathrm{Aut}(\mathfrak{e}_6)$ into four disjoint and exhausting classes:

Class 1, F contains an involution conjugate to σ_3;
Class 2, F doesn't contain any element conjugate to σ_3, but contains one conjugate to σ_4;
Class 3, $F \subset \mathrm{Int}(\mathfrak{e}_6)$ and it contains no Klein four subgroups conjugate to Γ_3;
Class 4, $F \subset \mathrm{Int}(\mathfrak{e}_6)$ and it contains a Klein four subgroup conjugate to Γ_3.

As $\mathrm{Int}(\mathfrak{e}_6)^{\sigma_3} \cong \mathrm{F}_4$ and $\mathrm{Int}(\mathfrak{e}_6)^{\sigma_4} \cong \mathrm{Sp}(4)/\langle(-I, -1)\rangle$, the classification for subgroups in Class 1 reduces to the classification in $\mathbf{F_4}$ case; the classification for subgroups in Class 2 reduces to the classification of subgroups of $\mathrm{Sp}(4)/\langle(-I, -1)\rangle$, but only those subgroups with any involution conjugate to $\mathbf{i}I$ or $\mathrm{diag}\{-I_2, I_2\}$ are concerned (cf. Sect. 6 for the reason). Our representatives of conjugacy classes in Class 1 are denoted as $\{F_{r,s} : r \leq 2, s \leq 3\}$ and representatives of conjugacy classes in Class 1 are denoted as $\{F_{\epsilon,\delta;r,s} : \epsilon+\delta \leq 1, r+s \leq 2\}$. Two important observations are: any subgroup in Class 3 is of the form $F \cap \mathrm{Int}(\mathfrak{e}_6)$ for a subgroup F in Class 1; and any subgroup in Class 4 is of the form $F \cap \mathrm{Int}(\mathfrak{e}_6)$ for a subgroup F in Class 2 satisfying some additional condition. Our representatives of conjugacy classes in Class 3 are denoted as $\{F'_{r,s} : r \leq 2, s \leq 3\}$ and representatives of conjugacy classes in Class 1 are denoted as $\{F'_{\epsilon,\delta;r,s} : \epsilon + \delta \leq 1, r + s \leq 2, s \geq 1\}$. In total, we have $3 \times 4 + 3 \times 6 + 3 \times 4 + 3 \times 3 = 51$ conjugacy classes of elementary abelian 2-subgroups.

In type $\mathbf{E_7}$, we divide the elementary abelian 2-subgroups F of $\mathrm{Aut}(\mathfrak{e}_7)$ into three disjoint and exhausting classes:

Class 1, F contains an involution conjugate to σ_2;
Class 2, F doesn't contain any element conjugate to σ_2, but contains one conjugate to σ_3;
Class 3, any involution in F is conjugate to σ_1.

From Table 2, we have that

$$\mathrm{Aut}(\mathfrak{e}_7)^{\sigma_2} \cong ((\mathrm{E}_6 \times \mathrm{U}(1))/\langle(c, e^{\frac{2\pi i}{3}})\rangle \rtimes \langle\omega\rangle,$$

where $1 \neq c \in Z_{\mathrm{E}_6}$, $\omega^2 = 1$, $(\mathfrak{e}_6 \oplus i\mathbb{R})^{\omega} = \mathfrak{f}_4 \oplus 0$, $\sigma_2 = (1, -1)$. Modulo U(1), we have a homomorphism $\pi : \mathrm{Aut}(\mathfrak{e}_7)^{\sigma_2} \longrightarrow \mathrm{Aut}(\mathfrak{e}_6)$. It turns out there is a bijection between conjugacy classes of elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{e}_7)$ in Class 1 and elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{e}_6)$. So we have fifty-one conjugacy classes in Class 1.

From Table 2, we have that

$$\mathrm{Aut}(\mathfrak{e}_7)^{\sigma_3} \cong (\mathrm{SU}(8)/\langle iI\rangle) \rtimes \langle\omega_0\rangle,$$

where $\omega_0^2 = 1,\ \omega_0 X \omega_0^{-1} = \overline{X}$ for any $X \in \mathrm{SU}(8),\ \sigma_3 = \frac{1+i}{\sqrt{2}} I$. So we have a homomorphism

$$\pi : \mathrm{Aut}(\mathfrak{e}_7)^{\sigma_3} \longrightarrow \mathrm{Aut}(\mathfrak{su}(8)) = (\mathrm{U}(8)/\mathrm{Z}_8) \rtimes \langle \omega_0 \rangle.$$

There is a bijection between conjugacy classes of elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{e}_7)$ in Class 2 and elementary abelian 2 subgroups of $\mathrm{Aut}(\mathfrak{su}(8))$ whose inner involutions are all conjugate to $I_{4,4} = \mathrm{diag}\{I_4, -I_4\}$ and outer involutions all conjugate to ω_0. These subgroups are classified by Propositions 2.4, 2.12 and Lemma 2.23. We get fourteen conjugacy classes in Class 2.

For an elementary abelian 2-subgroup F of $\mathrm{Aut}(\mathfrak{e}_7)$ in Class 3, we show either F is toral or it contains a rank 3 subgroup whose Klein four subgroups are all conjugate to Γ_1. In the first case, we can find an involution $\theta \in \mathrm{Aut}(\mathfrak{e}_7)^F$ such that elements in θF are all conjugate to σ_3. In the second case, we can find a Klein four subgroup $F' \subset \mathrm{Aut}(\mathfrak{e}_7)^F$ conjugate to Γ_6. Then F is a canonical subgroup of some well-chosen subgroup in Class 2 or Class 1. We get thirteen conjugacy classes in Class 3. In total, we have $51 + 14 + 13 = 78$ conjugacy classes of elementary abelian 2-subgroups.

In type $\mathbf{E_8}$, $\mathrm{Aut}(\mathfrak{e}_8) = \mathrm{E}_8$ has two conjugacy classes of involutions with representatives σ_1, σ_2. A nice observation is: for an elementary abelian 2-subgroup F of $\mathrm{Aut}(\mathfrak{e}_8)$ and any element x of F conjugate to σ_1, the subset

$$H_x = \{y \in F | xy \not\sim y\}$$

is a subgroup. We define H_F as the subgroup generated by elements of F conjugate to σ_1 and define

$$A_F = \{1\} \cup \{x \in F | x \sim \sigma_2,\ \text{and } \forall y \in F - \{1, x\}, xy \sim y\}.$$

Then $A_F \subset H_F$ if $H_F \neq 1$.

By [8, Table 6], we have that

$$\mathrm{Aut}(\mathfrak{e}_8)^{\Gamma_1} \cong ((\mathrm{E}_6 \times \mathrm{U}(1) \times \mathrm{U}(1))/\langle (c, e^{\frac{2\pi i}{3}}, 1) \rangle \rtimes \langle \omega \rangle,$$

where $1 \neq c \in Z_{\mathrm{E}_6},\ \omega^2 = 1,\ (\mathfrak{e}_6 \oplus i\mathbb{R} \oplus i\mathbb{R})^{\omega} = \mathfrak{f}_4 \oplus 0 \oplus 0,\ \Gamma_1 = \langle (1, 1, -1), (1, -1, 1) \rangle$. Modulo $\mathrm{U}(1) \times \mathrm{U}(1)$, we have a homomorphism $\pi : \mathrm{Aut}(\mathfrak{e}_8)^{\sigma_2} \longrightarrow \mathrm{Aut}(\mathfrak{e}_6)$. It turns out, π does not give a bijection between conjugacy classes of elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{e}_8)$ containing a Klein four subgroup conjugate to Γ_1 and elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{e}_6)$ and we find an explicit relation between these two kinds of conjugacy class and so get a classification of elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{e}_8)$ containing a Klein four subgroup conjugate to Γ_1. We have 48 conjugacy classes of elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{e}_8)$ containing a Klein four subgroup conjugate to Γ_1

When F doesn't contain any Klein four subgroup conjugate to Γ_1 and $H_F \neq 1$, we show that $\mathrm{rank}(H_F/A_F) = 1,\ \mathrm{rank} A_F \leq 3$ and $\mathrm{rank}(F/H_F) \leq 2$. Moreover, the conjugacy class of F is determined by the numbers $\mathrm{rank} A_F$ and $\mathrm{rank}(F/H_F)$. We have 12 conjugacy classes of elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{e}_8)$ of this type.

When $H_F = 1$, we have $\mathrm{rank} F \leq 5$ and the conjugacy class of F is determined by $\mathrm{rank} F$. So we have 6 conjugacy classes of elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{e}_8)$ of this type. In total, we have $48 + 12 + 6 = 66$ conjugacy classes of elementary abelian 2-subgroups.

3.5 Some notions

Definition 3.3 (Involution type) For an elementary abelian 2 subgroup F of a compact Lie group G, we call the distribution of conjugacy classes of involutions in F the *involution type* of F.

Definition 3.4 (Automizer group) For an elementary abelian 2 subgroup F of a compact Lie group G, we call $W(F) = N_G(F)/C_G(F)$ the automizer group of F.

$W(F)$ is also called Weyl group in Literature, e.g, [2]. The name of automizer is suggested by Professor R. Griess. We determine the automizer group $W(F)$ for each elementary abelian 2-subgroup F of $\mathrm{Aut}(\mathfrak{u}_0)$ with $\mathfrak{u}_0$ a compact exceptional simple Lie algebra. Conjugation action gives us an inclusion

$$W(F) \subset \mathrm{Aut}(F) = \mathrm{GL}(\mathrm{rank} F, \mathbb{F}_2).$$

Then we need to determine which automorphisms of F can be realized as $\mathrm{Ad}(g)$ for some $g \in G$.

We also introduce other notions like *translation subgroup*, *defect index*, *residual rank* in the following sections. As the definitions of these notions depend on the types of the Lie algebras (or Lie groups), we give the precise definitions in each section below. These notions help us to show the subgroups we constructed in different classes or in the same class but with different parameters are non-conjugate to each other.

4 G_2

For $G = \mathrm{Aut}(\mathfrak{g}_2)$, by Table 2 we know G has a unique conjugacy class of involution and we have $G^\sigma \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)/\langle(-1, -1)\rangle$ for any involution $\sigma \in G$.

Proposition 4.1 *The conjugacy class of an elementary abelian 2-subgroup F of G is determined by* $\mathrm{rank} F$ *and the possible values of* $\mathrm{rank} F$ *are* $\{0, 1, 2, 3\}$.

Proof We first prove that, for any $r \leq 3$, there exists a unique conjugacy class of ordered tuples $\{x_1, \ldots, x_r\}$ such that they generate an elementary abelian 2-subgroup of G with rank r. When $r = 1$, this follows from the classification of involutions in G, moreover we have

$$G^{x_1} \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)/\langle(-1, -1)\rangle$$

for any involution $x_1 \in G$. Let $x_2 \in G^{x_1}$ be an involution different from x_1. Then $x_2 \sim_{G^{x_1}} [(\mathbf{i}, \mathbf{i})]$. This proves the statement when $r = 2$. Moreover we have (when $x_2 = [(\mathbf{i}, \mathbf{i})]$, we can take $t = [(\mathbf{j}, \mathbf{j})]$ below)

$$G^{x_1,x_2} \cong ((\mathrm{U}(1) \times \mathrm{U}(1))/\langle(-1, -1)\rangle) \rtimes \langle t\rangle,$$

where $t^2 = 1$ and $t(z_1, z_2)t^{-1} = (z_1^{-1}, z_2^{-1})$, $\forall z_1, z_2 \in \mathrm{U}(1)$. Let $x_3 \in G^{x_1,x_2}$ be an involution not in $\langle x_1, x_2\rangle$. Then $x_3 \sim_{G^{x_1,x_2}} t$. This proves the statement when $r = 3$.

Moreover, we have $G^{x_1,x_2,x_3} = \langle x_1, x_2, x_3\rangle$, so $\langle x_1, x_2, x_3\rangle$ is not properly contained in any abelian subgroup of G. So an elementary abelian 2-subgroup F of G has rank at most 3. Then the proposition is proved. □

Corollary 4.2 *G has 4 conjugacy classes of an elementary abelian 2-subgroup.*

Proposition 4.3 *For $0 \leq r \leq 3$, for any elementary abelian 2-subgroup F_r of $G = \mathrm{G}_2$ with* $\mathrm{rank} F_r = r$*, we have $W(F_r) \cong \mathrm{GL}(r, \mathbb{F}_2)$.*

Proof This follows from the following statement: for any $r \leq 3$, there exists a unique conjugacy class of ordered tuples $\{x_1, \ldots, x_r\}$ such that they generate an elementary abelian 2-subgroup of G with rank r. This is proved during the above proof for Proposition 4.1. $\square$

5 $\mathbf{F_4}$

Let $G = \mathrm{Aut}(\mathfrak{f}_4)$. From Table 2, we see that G has two conjugacy classes of involutions with representatives σ_1, σ_2 such that

$$G^{\sigma_1} \cong \mathrm{Sp}(3) \times \mathrm{Sp}(1)/\langle(-I, -1)\rangle$$

and

$$G^{\sigma_2} \cong \mathrm{Spin}(9).$$

From [8, Page 18], we see that G^{σ_1} has three conjugacy classes of involutions except $\sigma_1 = (-I, 1) = (I, -1)$ with representatives $(\mathbf{i}I, \mathbf{i})$, $\left(\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, 1\right)$, $\left(\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, 1\right)$. Moreover in G, we have the conjugacy relations

$$(\mathbf{i}I, \mathbf{i}) \sim \sigma_1,$$
$$\left(\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, 1\right) \sim \sigma_1,$$
$$\left(\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, 1\right) \sim \sigma_2.$$

And G^{σ_2} has two conjugacy classes of involutions except $\sigma_2 = -1$ with representatives $e_1e_2e_3e_4,\ e_1e_2e_3e_4e_5e_6e_7e_8$. And in G, we have the conjugacy relations

$$e_1e_2e_3e_4 \sim \sigma_1$$

and

$$e_1e_2 \ldots e_8 \sim \sigma_2.$$

In $G^{\sigma_1} = \mathrm{Sp}(3) \times \mathrm{Sp}(1)/\langle(-I, -1)\rangle$, let $x_1 = (I, -1),\ x_2 = (\mathbf{i}I, \mathbf{i}),\ x_3 = (\mathbf{j}I, \mathbf{j})$,

$$x_4 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \quad x_5 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

For $0 \leq r \leq 2$ and $0 \leq s \leq 3$, define

$$F_{r,s} = \langle x_1, \ldots, x_s, x_4, \ldots, x_{3+r}\rangle$$

and $A_r = \langle x_4, \ldots, x_{3+r}\rangle$.

Definition 5.1 For an elementary abelian 2-subgroup $F \subset G$, define

$$A_F = \{x \in F : x \sim \sigma_2\} \cup \{1\}.$$

Proposition 5.2 *For an elementary abelian 2-subgroup F of G, A_F is a subgroup of F and we have* $\operatorname{rank} A_F \leq 2$ *and* $\operatorname{rank}(F/A_F) \leq 3$.

For each (r, s) with $0 \leq r \leq 2$ and $0 \leq s \leq 3$, there exists a unique conjugacy class of elementary abelian 2-subgroups F of G such that $\operatorname{rank} A_F = r$ *and* $\operatorname{rank}(F/A_F) = s$.

Proof Let $F \subset G$ be an elementary abelian 2-subgroup. By Table Table 3, we see that there are no Klein four subgroups of G with involutions type $(\sigma_1, \sigma_2, \sigma_2)$. Then for any distinct non-identity elements $x, y \in F$ with $x \sim y \sim \sigma_2$, we have $xy \sim \sigma_2$. So A_F is a subgroup.

In $G^{\sigma_2} = \mathrm{Spin}(9)$, besides $\sigma_2 = -1$, the elements conjugate to σ_2 in G are all conjugate to $e_1e_2\dots e_8$ in G^{σ_2}. There does not exist $x, y \in \mathrm{Spin}(9)$ with x, y, xy all conjugate to $e_1e_2\dots e_8$, so $\operatorname{rank} A_F \leq 2$.

In $G^{\sigma_1} = \mathrm{Sp}(3) \times \mathrm{Sp}(1)/\langle(-I, -1)\rangle$, the elements x with $x, \sigma_1 x = (-I, 1)x$ both conjugate to σ_1 in G are all conjugate to $[(\mathbf{i}I, \mathbf{i})]$ in G^{σ_1}. By this, it is clear that any elementary abelian 2-subgroup of G whose non-identity elements all conjugate to σ_1 has rank at most 3 ($\langle\sigma_1, [(\mathbf{i}I, \mathbf{i})], [(\mathbf{j}I, \mathbf{j})]\rangle$ is an example of rank 3). Since non-identity elements of a complement of A_F in F are all conjugate to σ_1, so $\operatorname{rank} F/A_F \leq 3$.

In $F = F_{r,s}$, we have $A_F = A_r$ is of rank r and $F/A_F = F_{r,s}/A_r$ is of rank s, so $F = F_{r,s}$ satisfies $\operatorname{rank} A_F = r$ and $\operatorname{rank} F/A_F = s$.

When $s = 0$, the uniqueness of the conjugacy class is showed in the proof for $r \leq 2$ above. When $s = 1$, we may and do assume that $\sigma_1 \in F$, then

$$F \subset G^{\sigma_1} = \mathrm{Sp}(3) \times \mathrm{Sp}(1)/\langle(-I, -1)\rangle.$$

The elements in G^{σ_1} which are conjugate to σ_2 in G are conjugate to $[(I_{2,1}, 1)]$ in G^{σ_1}. Moreover, any pair (x_1, x_2) with $x_1, x_2 \in G^{\sigma_1}$ are distinct and both conjugate to σ_2 in G is conjugate to $([(I_{2,1}, 1)], [(I_{1,2}, 1)])$ in G^{σ_1}. This proves the uniqueness of the conjugacy classes when $s = 1$. When $s = 2$, we may and do assume that $\sigma_1 \in F$ and $[(\mathbf{i}I, \mathbf{i})] \in F \cap G^{\sigma_1}$. Then

$$F \subset (G^{\sigma_1})^{[(\mathbf{i}I, \mathbf{i})]} = (\mathrm{U}(3) \times \mathrm{U}(1)/\langle(-I, -1)\rangle) \rtimes \langle t\rangle,$$

where $t = (\mathbf{j}I, \mathbf{j})$. Similarly as $s = 1$ case, we get the uniqueness of the conjugacy classes when $s = 2$. When $s = 3$, we may and do assume that $\sigma_1 \in F$ and $[(\mathbf{i}I, \mathbf{i})], [(\mathbf{j}I, \mathbf{j})] \in F \cap G^{\sigma_1}$. Then

$$F \subset (G^{\sigma_1})^{[(\mathbf{i}I, \mathbf{i})], [(\mathbf{j}I, \mathbf{j})]} = (\mathrm{SO}(3) \times \mathrm{SO}(1) \times \langle\sigma_1, [(\mathbf{i}I, \mathbf{i})], [(\mathbf{j}I, \mathbf{j})]\rangle.$$

We have $\mathrm{SO}(1) = 1$ and the elements in $\mathrm{SO}(3)$ which are conjugate to σ_2 in G are conjugate to $I_{2,1}$ in $\mathrm{SO}(3)$. Moreover any pair (x_1, x_2) with $x_1, x_2 \in \mathrm{SO}(3)$ are distinct and both conjugate to $I_{2,1}$ in $\mathrm{SO}(3)$ is conjugate to $(I_{2,1}, I_{1,2})$ in $\mathrm{SO}(3)$, so we get the uniqueness of the conjugacy classes when $s = 3$. □

Corollary 5.3 *We have 12 conjugacy classes of elementary abelian 2-subgroups in G.*

Proof Since $3 \times 4 = 12$, by Proposition 5.2, we get that there are 12 conjugacy classes of elementary abelian 2-subgroups in G. □

Proposition 5.4 *For two elementary abelian 2-subgroups $F, F' \subset G$, if $f : F \longrightarrow F'$ is an isomorphism such that $f(x) \sim x$, $\forall x \in F$, then there exists $g \in G$ such that $f = \mathrm{Ad}(g)$.*

Proof This is proved in the proof of Proposition 5.2 □

Proposition 5.5 *For any $r \leq 2$, $s \leq 3$, $W(F_{r,s}) \cong P(r, s, \mathbb{F}_2)$, where $P(r, s, \mathbb{F}_2)$ is the group of (r, s) block wise upper triangular matrices in $GL(r + s, \mathbb{F}_2)$.*

Proof For $F = F_{r,s}$, we have $A_F = A_r$ and any $g \in N_G(F)$ satisfies $gA_rg^{-1} = A_r$. By Proposition 5.4, we get $W(F) = N_G(F)/C_G(F) \cong P(r, s, \mathbb{F}_2)$. □

6 E_6

Let $G = \mathrm{Aut}(\mathfrak{e}_6)$. By Table 2, G has four conjugacy classes of involutions, two of them consist of inner automorphisms with representatives σ_1, σ_2 and the other two consist of outer automorphisms with representatives σ_3, σ_4. We have

$$(G_0)^{\sigma_1} \cong \mathrm{SU}(6) \times \mathrm{Sp}(1)/\langle (e^{\frac{2\pi i}{3}} I, 1), (-I, -1)\rangle,$$
$$(G_0)^{\sigma_2} \cong \mathrm{Spin}(10) \times \mathrm{U}(1)/\langle (c, i)\rangle, c = e_1 e_2 \ldots e_{10},$$
$$(G_0)^{\sigma_3} \cong \mathrm{F}_4$$

and

$$(G_0)^{\sigma_4} \cong \mathrm{Sp}(4)/\langle -I\rangle.$$

From [8, Page 15], we see that $(G_0)^{\sigma_1}$ has four conjugacy classes of involutions except σ_1. Their representatives and their conjugacy classes in G are as follows,

$$\left(\begin{pmatrix} -I_4 & \\ & I_2 \end{pmatrix}, 1\right) \sim \sigma_2, \quad \left(\begin{pmatrix} -I_2 & \\ & I_4 \end{pmatrix}, 1\right) \sim \sigma_1,$$
$$\left(\begin{pmatrix} iI_5 & \\ & -i \end{pmatrix}, i\right) \sim \sigma_2, \quad \left(\begin{pmatrix} iI_3 & \\ & -iI_3 \end{pmatrix}, i\right) \sim \sigma_1.$$

And $(G_0)^{\sigma_2}$ has four conjugacy classes of involutions except σ_2. Their representatives and their conjugacy classes in G are as follows,

$$\big(e_1 e_2 e_3 e_4, 1\big) \sim \sigma_1, \quad \big(e_1 e_2 \ldots e_8, 1\big) \sim \sigma_2,$$
$$\big(\Pi, \frac{1+i}{\sqrt{2}}\big) \sim \sigma_2, \quad \big(-\Pi, \frac{1+i}{\sqrt{2}}\big) \sim \sigma_1,$$

where

$$\Pi = \frac{1 + e_1 e_2}{\sqrt{2}} \frac{1 + e_3 e_4}{\sqrt{2}} \cdots \frac{1 + e_9 e_{10}}{\sqrt{2}}.$$

Definition 6.1 For an elementary abelian 2-subgroup $F \subset G$, define

$$\mu : F \cap G_0 \longrightarrow \{\pm 1\}$$

by $\mu(y) = -1$ if $y \sim \sigma_1$; and $\mu(y) = 1$ if $y \sim \sigma_2$.
And define

$$m : (F \cap G_0) \times (F \cap G_0) \longrightarrow \{\pm 1\}$$

by $m(y_1, y_2) = \mu(y_1 y_2)\mu(y_1)\mu(y_2)$.

Here m is not always a bilinear form.

Definition 6.2 For an elementary abelian 2-subgroup $F \subset G$, define the translation subgroup

$$A_F = \{x \in H \cap G_0 : \mu(x) = 1 \text{ and } m(x, y) = 1, \forall y \in F \cap G_0\}$$

and define the defect index

$$\mathrm{defe}(F) = |\{y \in F \cap G_0 : \mu(y) = 1\}| - |\{y \in F \cap G_0 : \mu(y) = -1\}|.$$

The subgroup A_F the has an equivalent definition as

$$A_F = \{1\} \cup \{x \in F | x \sim \sigma_2, \text{ and } y \sim xy \text{ for any } y \in F - \langle x\rangle, \},$$

this is why the name of "translation subgroup" arises.

6.1 Subgroups from F_4

In $(G_0)^{\sigma_3} \cong \mathrm{F}_4$, let τ_1, τ_2 be involutions such that

$$\mathfrak{f}_4^{\tau_1} \cong \mathfrak{sp}(3) \oplus \mathfrak{sp}(1), \quad \mathfrak{f}_4^{\tau_2} \cong \mathfrak{so}(9).$$

From [8, Page 15], we see that $\tau_1, \tau_2, \sigma_3\tau_1, \sigma_3\tau_2$ represent all conjugacy classes of involutions in G^{σ_3} except σ_3 and in we have the conjugacy relations in G,

$$\tau_1 \sim \sigma_1, \quad \tau_2 \sim \sigma_2,$$
$$\sigma_3\tau_1 \sim \sigma_4, \quad \sigma_3\tau_2 \sim \sigma_3.$$

We have $((G_0)^{\sigma_3})^{\tau_1} \cong \mathrm{Sp}(3) \times \mathrm{Sp}(1)/\langle(-I, -1)\rangle$. Let

$$x_0 = \sigma_3, \quad x_1 = \tau_1 = [(I, -1)],$$
$$x_2 = [(\mathbf{i}I, \mathbf{i})], \quad x_3 = [(\mathbf{j}I, \mathbf{j})],$$
$$x_4 = \left[\left(\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, 1\right)\right], \quad x_5 = \left[\left(\begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, 1\right)\right].$$

For a pair (r, s) with $r \leq 2$ and $s \leq 3$, define

$$F_{r,s} = \langle x_0, x_1, \ldots, x_s, x_4, \ldots, x_{3+r}\rangle$$

and

$$F'_{r,s} = \langle x_1, \ldots, x_s, x_4, \ldots, x_{3+r}\rangle.$$

Proposition 6.3 *For an elementary abelian 2-subgroup $F \subset G$, if F contains an element conjugate to σ_3, then $F \sim F_{r,s}$ for some (r, s) with $r \leq 2$, $s \leq 3$; if $F \subset G_0$ and it contains no Klein four subgroups conjugate to Γ_3, then $F \sim F'_{r,s}$ for some pair (r, s) with $r \leq 2$ and $s \leq 3$.*

Proof For the first statement, we may and do assume that $\sigma_3 \in F$, then $F \subset G^{\sigma_3} = F_4 \times \langle\sigma_3\rangle$. Then $F \sim F_{r,s}$ $(r \leq 2, s \leq 3)$ by Proposition 5.2.

For the latter statement, since we assume that F does not contain any Klein four subgroup conjugate to Γ_3, so F does not contain any Klein four subgroup of involutions type $(\sigma_1, \sigma_2, \sigma_2)$. Then we have $A_F = \{1\} \cup \{x \in F | x \sim \sigma_2\}$. Prove in the same line as the proof for Proposition 5.2, we can show that $\operatorname{rank} A_F \leq 2$, $\operatorname{rank}(F/A_F) \leq 3$ and the conjugacy class of F is uniquely determined by $\operatorname{rank} A_F$ and $\operatorname{rank}(F/A_F)$. Then we have $F \sim F'_{r,s}$ $(r \leq 2, s \leq 3)$ since $\operatorname{rank} A_{F'_{r,s}} = r$ and $\operatorname{rank}(F/A_{F'_{r,s}}) = s$. □

Lemma 6.4 *For an elementary abelian 2-subgroup F in Proposition 6.3,*

$$m(x, y) = -1 \Leftrightarrow x, y \in (F \cap G_0) - A_F, \quad \forall x, y \in F \cap G_0.$$

Proof This follows from the equality $A_F = \{1\} \cup \{x \in F | x \sim \sigma_2\}$. □

6.2 Subgroups from $\mathrm{Sp}(4)/\langle -I\rangle$

In $(G_0)^{\sigma_4} \cong \mathrm{Sp}(4)/\langle -I\rangle$, let $\tau_1 = \mathbf{i}I$, $\tau_2 = \begin{pmatrix} -I_2 & \\ & I_2 \end{pmatrix}$, $\tau_3 = \begin{pmatrix} -1 & \\ & I_3 \end{pmatrix}$. From [8, Pages 15–16], we see that $\tau_1, \tau_2, \tau_3, \sigma_4\tau_1, \sigma_4\tau_2, \sigma_4\tau_3$ represent all conjugacy classes of involutions in G^{σ_4} except σ_4 and we have the following conjugacy relations in G,

$$\tau_1 \sim \sigma_1, \ \tau_2 \sim \sigma_2, \ \tau_3 \sim \sigma_1$$

and

$$\sigma_4\tau_1 \sim \sigma_4,\ \sigma_4\tau_2 \sim \sigma_4,\ \sigma_4\tau_3 \sim \sigma_3.$$

Let $x_0 = \sigma_4,\ x_1 = \mathbf{i}I,\ x_2 = \mathbf{j}I$,

$$x_3 = \begin{pmatrix} -I_2 & \\ & I_2 \end{pmatrix}, x_4 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},$$

$$x_5 = \begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{pmatrix}, x_6 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}.$$

For any (ϵ, δ, r, s) with $\epsilon + \delta \leq 1, r + s \leq 2$, define

$$F_{\epsilon,\delta,r,s} = \langle x_0, x_1, \ldots, x_{\epsilon+2\delta}, x_3, \ldots, x_{r+2s} \rangle$$

and

$$F'_{\epsilon,\delta,r,s} = \langle x_1, \ldots, x_{\epsilon+2\delta}, x_3, \ldots, x_{r+2s} \rangle.$$

Proposition 6.5 *For an elementary abelian* 2*-subgroup $F \subset G$, if $F \not\subset G_0$ and it contains no elements conjugate to σ_3, then $F \sim F_{\epsilon,\delta,r,s}$ for some (ϵ, δ, r, s) with $\epsilon+\delta \leq 1$ and $r+s \leq 2$; if $F \subset G_0$ and it contains a Klein four subgroup conjugate to Γ_3, then $F \sim F'_{\epsilon,\delta,r,s}$ for some (ϵ, δ, r, s) with $\epsilon + \delta \leq 1,\ r + s \leq 2$ and $s \geq 1$.*

Proof For the first statement, we may assume that $\sigma_4 \in F$, then

$$F \cap G_0 \subset G_0^{\sigma_4} \cong \mathrm{Sp}(4)/\langle -I \rangle.$$

Any involution in $\mathrm{Sp}(4)/\langle -I \rangle$ is conjugate to one of

$$\tau_1 = [\mathbf{i}I], \tau_2 = [\mathrm{diag}\{I_2, -I_2\}], \tau_3 = [\mathrm{diag}\{1, -I_3\}].$$

Since $\sigma_4\tau_3 \sim \sigma_3$ in G and we assume that F contains no elements conjugate to σ_3, so any non-identity element of $F \cap G_0$ is conjugate to τ_1 or τ_2 in $\mathrm{Sp}(4)/\langle -I \rangle$. Then $F \cap G_0 \subset \mathrm{Sp}(4)/\langle -I \rangle$ is in the subclass discussed in Sect. 2.4. Then $F \sim F_{\epsilon,\delta,r,s}$ for some (ϵ, δ, r, s) with $\epsilon+\delta \leq 1$ and $r + s \leq 2$ by Proposition 2.24.

For the second statement, we may and do assume that $\Gamma_3 \subset F$, then

$$F \subset (G_0)^{\Gamma_3} \cong (\mathrm{U}(5) \times \mathrm{U}(1))/\langle (-I, -1), (e^{\frac{2\pi i}{3}}, 1) \rangle \cong (\mathrm{U}(5)/\langle e^{\frac{2\pi i}{3}} \rangle) \times \mathrm{U}(1).$$

Here we use that the map $(A, \lambda) \longmapsto (\lambda A, \lambda^2)$ gives an isomorphism

$$(\mathrm{U}(5) \times \mathrm{U}(1))/\langle (-I, -1) \rangle \cong \mathrm{U}(5) \times \mathrm{U}(1).$$

Since any abelian subgroup of $\mathrm{U}(5) \times \mathrm{U}(1)$ is total, so $F \subset G_0$ is total. We may and do assume that $F \subset \exp(\mathfrak{h}_0)$ for a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{u}_0 = \mathfrak{e}_6$. Choose a Chevelley involution θ of $\mathfrak{u}_0$ with respect to $\mathfrak{h}_0$. Then θ commutes with all elements $x \in \exp(\mathfrak{h}_0)$ satisfying $x^2 = 1$. Moreover, we have $\theta \sim \sigma_4$ (since $\dim \mathfrak{u}_0^\theta = 63$) and $\theta x \sim \theta$ for any $x \in \exp(\mathfrak{h}_0)$. Then $\langle F, \theta \rangle$ is an elementary abelian 2-subgroup without elements conjugate to σ_3. By the first statement, we get that $\langle F, \theta \rangle \sim F_{\epsilon,\delta,r,s}$ for some (ϵ, δ, r, s) with $\epsilon + \delta \leq 1$ and $r + s \leq 2$. Then $F \sim F'_{\epsilon,\delta,r,s}$. Since we assume that F contains a Klein four subgroup conjugate to Γ_3, so we have $s \geq 1$. □

Let F be an elementary abelian 2-subgroup of G without elements conjugate to σ_3 and containing an element conjugate to σ_4. For any $x \in F$ with $x \sim \sigma_4$, we have $F \cap G_0 \subset (G_0)^{\sigma_4} \cong \mathrm{Sp}(4)/\langle -I \rangle$. With this inclusion we have a function $\mu_x : F \cap G_0 \longrightarrow \{\pm 1\}$ and a map $m_x : (F \cap G_0) \times (F \cap G_0) \longrightarrow \{\pm 1\}$ (cf. Sect. 2.2).

Lemma 6.6 *We have $\mu_x = \mu$ and $m_x = m$.*

Proof We may assume that $x = \sigma_4$, then $F \cap G_0 \subset (G_0)^{\sigma_4} \cong \mathrm{Sp}(4)/\langle -I \rangle$. Since F does not have any element conjugate to σ_3, from the proof for Proposition 6.5 we see that any element of $F \cap G_0$ is conjugate to $\tau_1 = \mathbf{i}I$ or $\tau_2 = \begin{pmatrix} -I_2 & \\ & I_2 \end{pmatrix}$ in $(G_0)^{\sigma_4} \cong \mathrm{Sp}(4)/\langle -I \rangle$. Since $\tau_1 \sim_G \sigma_1$ and $\tau_2 \sim_G \sigma_2$, so we have $\mu_x = \mu$. Then we have $m_x = m$ as well. □

6.3 Automizer groups

Proposition 6.7 *We have the following formulas for* $\mathrm{rank} A_F$ *and* $\mathrm{defe} F$*,*

(1) *for* $F = F_{r,s}$, $r \leq 2$, $s \leq 3$, $\mathrm{rank} A_F = r$, $\mathrm{defe} F = 2^r(2 - 2^s)$*;*
(2) *for* $F = F'_{r,s}$, $r \leq 2$, $s \leq 3$, $\mathrm{rank} A_F = r$, $\mathrm{defe} F = 2^r(2 - 2^s)$*;*
(3) *for* $F = F_{\epsilon,\delta,r,s}$, $\epsilon + \delta \leq 1$, $r + s \leq 2$, $\mathrm{rank} A_F = r$, $\mathrm{defe} F = (1 - \epsilon)(-1)^{\delta} 2^{r+s+\delta}$*;*
(4) *for* $F = F'_{\epsilon,\delta,r,s}$, $\epsilon + \delta \leq 1$, $r + s \leq 2$, $s \geq 1$, $\mathrm{rank} A_F = r$, $\mathrm{defe} F = (1 - \epsilon)(-1)^{\delta} 2^{r+s+\delta}$.

Proof They follow from Lemmas 6.4 and 6.6. □

Corollary 6.8 *We have 51 conjugacy classes of elementary abelian 2-subgroups of* $\mathrm{Aut}(\mathfrak{e}_6)$.

Proof By the formulas of $\mathrm{rank} A_F$ and $\mathrm{defe} F$ in Proposition 6.7, we see that the subgroups in each family with different parameters are non-conjugate. And the subgroups in different families are clearly non-conjugate, so these subgroups are non-conjugate to each other. In total, we have $3 \times 4 + 3 \times 4 + 3 \times 6 + 3 \times 3 = 51$ conjugacy classes. □

Proposition 6.9 *For two elementary abelian 2-subgroups $F, F' \subset G$, if an isomorphism $f : F \longrightarrow F'$ has the property that $f(x) \sim x$ for any $x \in F$, then $f = \mathrm{Ad}(g)$ for some $g \in G$.*

Proof We may and do assume that $F = F'$ and they are equal to one of

$$F_{r,s},\ F'_{r,s},\ F_{\epsilon,\delta,r,s},\ F'_{\epsilon,\delta,r,s}.$$

When $F = F' = F_{r,s}$, we may and do assume that $f(\sigma_3) = \sigma_3$, then $F \cap G_0 = F' \cap G_0 \subset (G_0)^{\sigma_3} = F_4$. By the proof of Proposition 5.2, we get some $g \in (G_0)^{\sigma_3}$ such that $f = \mathrm{Ad}(g)$.

When $F = F' = F_{r,s}$, similar as the proof for Proposition 5.2, we find some $g \in G_0$ such that $f = \mathrm{Ad}(g)$.

When $F = F' = F_{\epsilon,\delta,r,s}$, we may and do assume that $f(\sigma_4) = \sigma_4$, then $F \cap G_0 = F' \cap G_0 \subset (G_0)^{\sigma_4} = \mathrm{Sp}(4)/\langle -I \rangle$ and non-identity elements of $F \cap G_0 = F' \cap G_0$ are all conjugate to $\mathbf{i}I$ or $[I_{2,2}]$ in $\mathrm{Sp}(4)/\langle -I \rangle$. Then $f = \mathrm{Ad}(g)$ for some $g \in G_0^{\sigma_4}$ by Proposition 2.24.

When $F = F' = F'_{\epsilon,\delta,r,s}$, since $F'_{\epsilon,\delta,r,s} \subset (G_0)^{\sigma_4} = \mathrm{Sp}(4)/\langle -I \rangle$ and non-identity elements of $F = F'$ are all conjugate to $\mathbf{i}I$ or $[I_{2,2}]$ in $\mathrm{Sp}(4)/\langle -I \rangle$. Then $f = \mathrm{Ad}(g)$ for some $g \in G_0^{\sigma_4}$ by Proposition 2.24. □

Proposition 6.10 *We have the following description for the automizer groups,*

(1) $r \leq 2$, $s \leq 3$, $W(F_{r,s}) \cong (\mathbb{F}_2)^r \rtimes P(r, s, \mathbb{F}_2)$*;*

(2) $r \leq 2,\ s \leq 3,\ W(F'_{r,s}) \cong P(r, s, \mathbb{F}_2)$;
(3) $\epsilon + \delta \leq 1,\ r + s \leq 2$,

$$W(\mathbb{F}_{\epsilon,\delta,r,s}) \cong \mathbb{F}_2^{r+2s+\epsilon+2\delta} \rtimes \big(\mathrm{Hom}(\mathbb{F}_2^{\epsilon+2\delta+2s}, \mathbb{F}_2^r) \rtimes (\mathrm{GL}(r, \mathbb{F}_2) \times \mathrm{Sp}(s; \epsilon, \delta))\big);$$

(4) $\epsilon + \delta \leq 1,\ r + s \leq 2,\ s \geq 1$,

$$W(F'_{\epsilon,\delta,r,s}) \cong \mathrm{Hom}(\mathbb{F}_2^{\epsilon+2\delta+2s}, \mathbb{F}_2^r) \rtimes (\mathrm{GL}(r, \mathbb{F}_2) \times \mathrm{Sp}(s; \epsilon, \delta)).$$

Proof The action of any $w \in W(F)$ preserves μ and m on $F \cap G_0$ and the conjugacy classes of elements in $F - (F \cap G_0)$. By Proposition 6.9, an automorphism of F preserves these data is actually the action of some $w \in W(F)$ on F. Then by Lemmas 6.4 and 6.6, we get these automizer groups. □

7 E_7

Let $G = \mathrm{Aut}(\mathfrak{e}_7)$. By Table 2 we see that there are three conjugacy classes of involutions in G with representatives $\sigma_1, \sigma_2, \sigma_3$ and we have

$$G^{\sigma_1} \cong (\mathrm{Spin}(12) \times \mathrm{Sp}(1))/\langle (c, 1), (-c, -1)\rangle,$$
$$G^{\sigma_2} \cong ((\mathrm{E}_6 \times \mathrm{U}(1))/\langle (c', e^{\frac{2\pi i}{3}})\rangle) \rtimes \langle \omega \rangle,$$
$$G^{\sigma_3} \cong (\mathrm{SU}(8)/\langle iI \rangle) \rtimes \langle \omega \rangle,$$

where $c = e_1 e_2 \dots e_{12},\ 1 \neq c' \in Z_{E_6},\ \omega^2 = 1$, and

$$(\mathfrak{e}_6 \oplus i\mathbb{R})^{\omega} = \mathfrak{f}_4 \oplus 0,\ \mathfrak{su}(8)^{\omega} \cong \mathfrak{sp}(4).$$

Definition 7.1 For an elementary abelian 2-subgroup F of G, define

$$H_F = \{1\} \cup \{x \in F | x \sim \sigma_1\};$$

define

$$m : H_F \times H_F \longrightarrow \{\pm 1\}$$

by $m(x, y) = -1$ if $\langle x, y\rangle \sim \Gamma_1$, and $m(x, y) = 1$ otherwise.

Definition 7.2 Define the translation subgroup

$$A_F := \{x \in H_F : \forall y \in F - H_F, y \sim xy;\ \text{and}\ \forall y \in H_F, m(x, y) = 1\}$$

and the defect index

$$\mathrm{defe}(F) = |\{x \in F : x \sim \sigma_2\}| - |\{x \in F : x \sim \sigma_3\}|.$$

For any $x \in F$ with $x \sim \sigma_2$, let

$$H_x = \{y \in H_F | xy \sim \sigma_2\},$$

which is not always a subgroup.

Lemma 7.3 *H_F is a subgroup of F and we have* $\mathrm{rank}(F/H_F) \leq 2$.

Proof Since the product of any two distinct elements in F conjugate to σ_1 is also conjugate to σ_1, so H_F is a subgroup.

Suppose that $\mathrm{rank}(F/H_F) \geq 3$, then there exists a rank 3 subgroup $F' \subset F$ with $H_{F'} = 1$. For any $1 \neq x \in F'$, $G^x \sim G^{\sigma_2}$ or G^{σ_3} has only two connected components, so $\mathrm{rank}(F' \cap (G^x)_0) \geq 2$. Choose $y \in F' \cap (G^x)_0 - \langle x \rangle$, then $\langle x, y \rangle$ is a toral Klein four subgroup of G. By Table 3, at least one of x, y, xy is conjugate to σ_1, which contradicts that $H_{F'} = 1$. □

7.1 Subgroups from E_6

By Table 2, we have that

$$G^{\sigma_2} \cong \big((\mathrm{E}_6 \times \mathrm{U}(1))/\langle (c, e^{\frac{2\pi i}{3}}) \rangle\big) \rtimes \langle \omega \rangle,$$

where $\omega^2 = 1$ and $(\mathfrak{e}_6 \oplus i\mathbb{R})^{\omega} = \mathfrak{f}_4 \oplus 0$. Let $\tau_1, \tau_2 \in \mathrm{E}_6$ be involutions with

$$\mathfrak{e}_6^{\tau_1} \cong \mathfrak{su}(6) \oplus \mathfrak{sp}(1), \ \ \mathfrak{e}_6^{\tau_2} \cong \mathfrak{so}(10) \oplus i\mathbb{R}.$$

Let $\eta_1, \eta_2 \in \mathrm{E}_6^{\omega} \cong \mathrm{F}_4$ be involutions with

$$\mathfrak{f}_4^{\eta_1} \cong \mathfrak{sp}(3) \oplus \mathfrak{sp}(1), \mathfrak{f}_4^{\eta_2} \cong \mathfrak{so}(9).$$

Let $\tau_3 = \omega, \ \tau_4 = \eta_1\omega$. From [8, Page 16], we see that $\tau_1, \ \tau_2, \ \sigma_2\tau_1, \ \sigma_2\tau_2, \ \tau_3, \ \tau_4$ represent all conjugacy classes of involutions in G^{σ_2} except σ_2 and we have the following conjugacy relations in G,

$$\begin{aligned}
&\tau_1 \sim \tau_2 \sim \sigma_1, \\
&\sigma_2\tau_1 \sim \sigma_3, \sigma_2\tau_2 \sim \sigma_2, \\
&\tau_3 \sim \sigma_2\tau_3 \sim \sigma_2, \\
&\tau_4 \sim \sigma_2\tau_4 \sim \sigma_3.
\end{aligned}$$

Lemma 7.4 *In G^{σ_2}, we have the conjugacy relations*

$$\eta_1 \sim_{\mathrm{E}_6} \tau_1, \ \ \eta_2 \sim_{\mathrm{E}_6} \tau_2, \ \ \eta_2\omega \sim_{\mathrm{E}_6} \omega.$$

Proof This follows from [8, Page 15] (for $\mathrm{Aut}(\mathfrak{e}_6)^{\sigma_3}$), the elements $\tau_1, \ \tau_2, \ \omega, \ \eta_1\omega, \ \eta_1, \ \eta_2$ correspond to the elements $\sigma_1, \ \sigma_2, \ \sigma_3, \ \sigma_4, \ \tau_1, \ \tau_2$ there. □

Let L_1, L_2, L_3, L_4 be Klein four subgroups of E_6 of involution types (τ_1, τ_1, τ_1), (τ_1, τ_1, τ_2), (τ_1, τ_2, τ_2), (τ_2, τ_2, τ_2) respectively. By Table 3, we have that

$$\begin{aligned}
&(\mathfrak{e}_7^{\sigma_2})^{L_1} \cong \mathfrak{su}(3)^2 \oplus (i\mathbb{R})^3, \\
&(\mathfrak{e}_7^{\sigma_2})^{L_2} \cong \mathfrak{su}(4) \oplus \mathfrak{su}(2)^2 \oplus (i\mathbb{R})^2, \\
&(\mathfrak{e}_7^{\sigma_2})^{L_3} \cong \mathfrak{su}(5) \oplus (i\mathbb{R})^3, \\
&(\mathfrak{e}_7^{\sigma_2})^{L_4} \cong \mathfrak{so}(8) \oplus (i\mathbb{R})^3.
\end{aligned}$$

Lemma 7.5 *In G, we have $L_1 \sim L_3 \sim \Gamma_1$ and $L_2 \sim L_4 \sim \Gamma_2$.*

Proof First since $\tau_1 \sim \tau_2 \sim \sigma_1$ in G, so each of L_1, L_2, L_3, L_4 is conjugate to Γ_1 or Γ_2. Since $\mathfrak{su}(3)^2 \oplus (i\mathbb{R})^3$, $\mathfrak{su}(5) \oplus (i\mathbb{R})^3$ are not symmetric subalgebras of $\mathfrak{e}_7^{\Gamma_2} \cong \mathfrak{so}(8) \oplus \mathfrak{su}(2)^3$ and $\mathfrak{su}(4) \oplus \mathfrak{su}(2)^2 \oplus (i\mathbb{R})^2$, $\mathfrak{so}(8) \oplus (i\mathbb{R})^3$ are not symmetric subalgebras of $\mathfrak{e}_7^{\Gamma_1} \cong \mathfrak{su}(6) \oplus (i\mathbb{R})^2$, so L_1, L_3 are conjugate to Γ_1 and L_2, L_4 are conjugate to Γ_2. □

Let $F \subset G$ be an elementary abelian 2-subgroup containing an element conjugate to σ_2, we may an do assume that $\sigma_2 \in F$, then

$$F \subset G^{\sigma_2} \cong ((\mathrm{E}_6 \times \mathrm{U}(1))/\langle(c, e^{\frac{2\pi i}{3}})\rangle) \rtimes \langle\omega\rangle,$$

where c is a non-trivial central element of E_6, $c^3 = 1$, $\omega^2 = 1$ and $(\mathfrak{e}_6 \oplus i\mathbb{R})^{\omega} = \mathfrak{f}_4 \oplus 0$. Let $G_{\sigma_2} = (\mathrm{E}_6 \times 1) \ltimes \langle\omega\rangle$ be the subgroup generated by E_6 $(= \mathrm{E}_6 \times 1)$ and ω. This definition of G_{σ_2} is not quite canonical, another choice is to define it as $(\mathrm{E}_6 \times 1) \ltimes \langle\sigma_2\omega\rangle$, but these are conjugate since

$$\begin{aligned}(1, i)\omega(1, i)^{-1} &= \omega(\omega^{-1}(1, i)\omega)(1, i)^{-1}\\ &= \omega(1, -i)(1, -i) = \omega(1, -1)\\ &= \omega\sigma_2.\end{aligned}$$

And so they are equivalent,

Lemma 7.6 *For an elementary abelian 2-subgroup $F \subset G$ containing σ_2, in the inclusion $F \subset G^{\sigma_2} \cong ((\mathrm{E}_6 \times \mathrm{U}(1))/\langle(c, e^{\frac{2\pi i}{3}})\rangle) \rtimes \langle\omega\rangle$, we have $H_F = F \cap \mathrm{E}_6$.*

Moreover, the map $m : H_F \times H_F \longrightarrow \{\pm 1\}$ is equal to the the similar map when H_F is viewed as a subgroup of E_6 (or $\mathrm{E}_6/\langle c\rangle = \mathrm{Int}(\mathfrak{e}_6)$).

Proof $H_F = F \cap \mathrm{E}_6$ follows from the comparison of conjugacy classes of involutions in G^{σ_2} and in G. The two maps m are equal follows from Lemma 7.5. □

Let $\pi : G_{\sigma_2} \longrightarrow \mathrm{Aut}(\mathfrak{e}_6)$ be the adjoint homomorphism and $p : G_{\sigma_2} \longrightarrow G^{\sigma_2}$ be the inclusion map. For any elementary abelian 2-subgroup K of $\mathrm{Aut}(\mathfrak{e}_6)$, $p(\pi^{-1}K) \times \langle\sigma_2\rangle$ is the direct product of its unique Sylow 2-subgroup F and $\langle(c, 1)\rangle$. Let

$$\begin{aligned}&\{F_{r,s} : r \leq 2, s \leq 3\},\\ &\{F'_{r,s} : r \leq 2, s \leq 3\},\\ &\{F_{\epsilon,\delta,r,s} : \epsilon + \delta \leq 1, r + s \leq 2\},\\ &\{F'_{\epsilon,\delta,r,s} : \epsilon + \delta \leq 1, r + s \leq 2, s \geq 1\}\end{aligned}$$

be elementary abelian 2-subgroups of $G^{\sigma_2} \subset G$ obtained from elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{e}_6)$ with the corresponding notation in this way.

Proposition 7.7 *Any elementary abelian 2-subgroup of G with an element conjugate to σ_2 is conjugate to one of $F_{r,s}$, $F'_{r,s}$, $F_{\epsilon,\delta,r,s}$, $F'_{\epsilon,\delta,r,s}$.*

Proof We may and do assume that $\sigma_2 \in F$, then

$$F \subset G^{\sigma_2} \cong ((\mathrm{E}_6 \times U(1))/\langle(c, e^{\frac{2\pi i}{3}})\rangle) \rtimes \langle\omega\rangle.$$

By Lemma 7.3, we have $\mathrm{rank}(F/H_F) \leq 2$. When $\mathrm{rank}(F/H_F) = 1$, we have $F \subset \mathrm{E}_6 \times \langle\sigma_2\rangle$. When $\mathrm{rank}(F/H_F) = 2$, we may an do assume that $\omega \in F$ or $\tau_4 = \eta_1\omega \in F$, then $F \subset (\mathrm{E}_6 \rtimes \langle\omega\rangle) \times \langle\sigma_2\rangle$. Then the conclusion follows from Propositions 6.3 and 6.5. □

Proposition 7.8 *The four families have the following characterization, so subgroups in different families are not conjugate to each other.*

(1) *F is conjugate to some $F_{r,s}$ if and only if F contains a subgroup conjugate to Γ_6;*
(2) *F is conjugate to some $F'_{r,s}$ if and only if $\mathrm{rank}(F/H_F) = 1$, F contains an element x conjugate to σ_2 and H_x is a subgroup;*

(3) *F is conjugate to some $F_{\epsilon,\delta,r,s}$ if and only if* $\mathrm{rank}(F/H_F) = 2$, *F contains an element conjugate to σ_2 but contains no subgroups conjugate to Γ_6;*
(4) *F is conjugate to some $F'_{\epsilon,\delta,r,s}$ if and only if* $\mathrm{rank}(F/H_F) = 1$, *F contains an element conjugate to σ_2 and H_x is not a subgroup.*

Proof (1) and (3) are clear. (2) and (4) follow from the comparison of conjugacy classes of involutions in G^{σ_2} and in G and the classification of elementary abelian 2-subgroups of $\mathrm{Int}(\mathfrak{e}_6)$ (by Propositions 6.3 and 6.5). □

We make a remark that, for a subgroup F in case (2) or (4), if H_x for some $x \in F$ with $x \sim \sigma_2$ is an subgroup, then the $H_{x'}$ for any other $x' \in F$ with $x' \sim \sigma_2$ is a subgroup; conversely, if H_x for $x \in F$ with $x \sim \sigma_2$ is not a subgroup, then the $H_{x'}$ for any other $x' \in F$ with $x' \sim \sigma_2$ is not a subgroup.

Proposition 7.9 *We have the following formulas for* $\mathrm{rank} A_F$ *and* $\mathrm{defe} F$.

(1) *For* $F = F_{r,s}$, $r \leq 2$, $s \leq 3$, $\mathrm{rank} A_F = r$, $\mathrm{defe} F = 3 \cdot 2^r(2 - 2^s)$;
(2) *For* $F = F'_{r,s}$, $r \leq 2$, $s \leq 3$, $\mathrm{rank} A_F = r$, $\mathrm{defe} F = 2^r(2 - 2^s)$;
(3) *For* $F = F_{\epsilon,\delta,r,s}$, $\epsilon + \delta \leq 1$, $r + s \leq 2$, $\mathrm{rank} A_F = r$, $\mathrm{defe} F = (1-\epsilon)(-1)^{\delta} 2^{r+s+\delta} - 2^{1+r+\epsilon+2s+2\delta}$;
(4) *For* $F = F'_{\epsilon,\delta,r,s}$, $\epsilon + \delta \leq 1$, $r + s \leq 2$, $s \geq 1$, $\mathrm{rank} A_F = r$, $\mathrm{defe} F = (1-\epsilon)(-1)^{\delta} 2^{r+s+\delta}$.

Proof These follows from Lemma 7.6 and Proposition 6.7. □

Proposition 7.10 *Any two of the subgroups* $\{F_{r,s}\}$, $\{F'_{r,s}\}$, $\{F_{\epsilon,\delta,r,s}\}$, $\{F'_{\epsilon,\delta,r,s}\}$ *are non-conjugate.*

Proof This follows from Propositions 7.7 and 7.9. □

7.2 Subgroups from SU(8) or SO(8)

By Table 2, we have that

$$G^{\sigma_3} \cong (\mathrm{SU}(8)/\langle iI \rangle) \rtimes \langle \omega \rangle,$$

where $\omega^2 = 1$, $(\mathfrak{u}_0^{\sigma_3})^{\omega} = \mathfrak{sp}(4)$ and $\mathfrak{p} \cong \wedge^4(\mathbb{C}^8)$. Let $\tau_1 = [I_{2,6}]$ and $\tau_2 = [I_{4,4}]$.

Let $\omega_0 = \omega \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}$. Then $\omega_0^2 = 1$ and $(\mathrm{SU}(8)/\langle iI \rangle)^{\omega_0} = (\mathrm{SO}(8)/\langle -I \rangle) \times \langle \sigma_3 \rangle$. In $((\mathrm{SU}(8)/\langle iI \rangle)^{\omega_0})_0 = \mathrm{SO}(8)/\langle -I \rangle$, let

$$\eta_1 = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} -I_4 & \\ & I_4 \end{pmatrix},$$
$$\eta_3 = \begin{pmatrix} -I_2 & \\ & I_6 \end{pmatrix}, \quad \eta_4 = \begin{pmatrix} 0 & I_{1,3} \\ -I_{1,3} & 0 \end{pmatrix},$$

where $I_{1,3} = \mathrm{diag}\{-1, 1, 1, 1\}$. Let $\tau_3 = \omega_0$, $\tau_4 = \eta_1\omega_0$. From [8, Pages 16–17], we see that τ_1, τ_2, $\sigma_3\tau_1$, $\sigma_3\tau_2$, τ_3, τ_4, $\sigma_3\tau_4$ represent all conjugacy classes of involutions in G^{σ_3} except $\sigma_3 = \frac{1+i}{\sqrt{2}} I$ and we have the following conjugacy relations in G,

$$\begin{aligned}
&\tau_1 \sim \tau_2 \sim \sigma_1,\\
&\sigma_3\tau_1 \sim \sigma_2,\ \sigma_3\tau_2 \sim \sigma_3,\\
&\tau_3 \sim \sigma_3,\ \tau_4 \sim \sigma_2,\\
&\tau_4\sigma_3 \sim \sigma_3.
\end{aligned}$$

Lemma 7.11 *In G^{σ_3}, we have the conjugacy classes*

$$\begin{aligned}
&\eta_3 \sim_{G^{\sigma_3}} \tau_1,\\
&\eta_1 \sim_{G^{\sigma_3}} \eta_2 \sim_{G^{\sigma_3}} \eta_4 \sim_{G^{\sigma_3}} \tau_2,\\
&\eta_2\omega_0 \sim_{G^{\sigma_3}} \eta_3\omega_0 \sim_{G^{\sigma_3}} \omega_0 = \tau_3,\\
&\eta_4\omega_0 \sim_{G^{\sigma_3}} \eta_1\omega_0\sigma_3 = \tau_4\sigma_2.
\end{aligned}$$

Proof These conjugacy relations can be prove by matrix calculation in the group $(\mathrm{SU}(8)/\langle iI\rangle) \rtimes \langle\omega_0\rangle$. We show the relation $\eta_4\omega_0 \sim_{G^{\sigma_3}} \eta_1\omega_0\sigma_3$ here, which is the most complicated one among them.

Let $y = e^{\frac{\pi i}{8}}\mathrm{diag}\{I_7, -1\} \in \mathrm{SU}(8)/\langle iI\rangle$, then

$$\begin{aligned}
y(\eta_4\omega_0)y^{-1} &= (y\eta_4 y^{-1})\omega_0(\omega_0^{-1}y\omega_0)y^{-1}\\
&= \eta_1\omega_0 y^{-1}y^{-1} = \eta_1\omega_0 e^{\frac{-\pi i}{4}}\\
&= \eta_1\omega_0\sigma_2,
\end{aligned}$$

in the last equality we use $e^{\frac{-\pi i}{4}}I = (e^{\frac{\pi i}{4}}I)(iI)^{-1} = e^{\frac{\pi i}{4}}I = \sigma_3$ in $\mathrm{SU}(8)/\langle iI\rangle$. □

Let

$$\begin{aligned}
M_1 &= \left\langle \begin{pmatrix} -I_4 & \\ & I_4 \end{pmatrix}, \begin{pmatrix} 0_4 & I_4 \\ I_4 & 0_4 \end{pmatrix} \right\rangle,\\
M_2 &= \langle \mathrm{diag}\{-I_4, I_4\}, \mathrm{diag}\{-I_2, I_2, -I_2, I_2\}\rangle,
\end{aligned}$$

then $(\mathfrak{e}_7^{\sigma_3})^{M_1} \cong \mathfrak{su}(4)$ and $(\mathfrak{e}_7^{\sigma_3})^{M_2} \cong (\mathfrak{sp}(1))^4 \oplus (i\mathbb{R})^3$.

Lemma 7.12 *In G, we have $M_1 \sim \Gamma_1$ and $M_2 \sim \Gamma_2$.*

Proof First since M_1, M_2 are pure σ_1 subgroups, so each of them is conjugate to Γ_1 or Γ_2. Since $\mathfrak{su}(4)$ is not a symmetric subalgebra of $\mathfrak{e}_7^{\Gamma_2} \cong \mathfrak{so}(8) \oplus (\mathfrak{sp}(1))^3$ and $(\mathfrak{sp}(1))^4 \oplus (i\mathbb{R})^3$ is not a symmetric subalgebra of $\mathfrak{e}_7^{\Gamma_1} \cong \mathfrak{su}(6) \oplus (i\mathbb{R})^2$, so we have $M_1 \sim \Gamma_1$ and $M_2 \sim \Gamma_2$. □

Let F be an elementary abelian 2-subgroup of G with $\sigma_3 \in F$. Then

$$F \subset G^{\sigma_3} \cong (\mathrm{SU}(8)/\langle iI\rangle) \rtimes \langle\omega_0\rangle,$$

where $\omega_0^2 = 1$ and $\mathfrak{su}(8)^{\omega_0} = \mathfrak{so}(8)$. If F has no elements conjugate to σ_2, from the description of conjugacy classes of involutions in G^{σ_3} as above, we get that $x \sim \tau_2 = \mathrm{diag}\{-I_4, I_4\}$ for any $1 \neq x \in H_F$.

Lemma 7.13 *For an elementary abelian 2-subgroup F of G containing σ_3 and without elements conjugate to σ_2, in the inclusion $F \subset G^{\sigma_3} \cong (\mathrm{SU}(8)/\langle iI\rangle) \rtimes \langle\omega_0\rangle$, we have $H_F \subset \mathrm{SU}(8)/\langle iI\rangle$ and the homomorphism*

$$H_F \longrightarrow \mathrm{SU}(8)/\langle iI, \sigma_3\rangle = \mathrm{SU}(8)\Big/\left\langle \frac{1+i}{\sqrt{2}}I \right\rangle = \mathrm{PSU}(8)$$

is injective. Moreover, the map

$$m : H_F \times H_F \longrightarrow \{\pm 1\}$$

is equal to the similar map when H_F is regarded as a subgroup of PSU(8).

Proof We have $H_F \subset \mathrm{SU}(8)/\langle iI \rangle$ since any involution in $\omega_0 \mathrm{SU}(8)/\langle iI \rangle$ is conjugate to σ_2 or σ_3. The map $H_F \longrightarrow \mathrm{PSU}(8)$ is injective since $\sigma_3 \notin H_F$. The two maps m are equal follows from Lemma 7.12. □

In $(G^{\sigma_3})_0 \cong \mathrm{SU}(8)/\langle iI \rangle$, let $y_1 = \mathrm{diag}\{-I_4, I_4\}$,

$$\begin{aligned}
y_2 &= \mathrm{diag}\{-I_2, I_2, -I_2, I_2\},\\
y_3 &= \mathrm{diag}\{-1, 1, -1, 1, -1, 1, -1, 1\},\\
y_4 &= \begin{pmatrix} 0_4 & I_4 \\ I_4 & 0_4 \end{pmatrix},\\
y_5 &= \begin{pmatrix} 0_2 & I_2 & & \\ I_2 & 0_2 & & \\ & & 0_2 & I_2 \\ & & I_2 & 0_2 \end{pmatrix},\\
y_6 &= \begin{pmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & 0 & 1 & & \\ & & & & 1 & 0 & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{pmatrix}.
\end{aligned}$$

For each (r, s) with $r + s \leq 3$, let $F''_{r,s} = \langle \sigma_3, y_1, y_2, \ldots, y_{r+s}, y_4, \ldots, y_{3+s} \rangle$.

In $(G^{\sigma_3})^{\omega_0} = (\mathrm{SO}(8)/\langle -I \rangle) \times \langle \sigma_3, \omega_0 \rangle$, let $x_1 = \mathrm{diag}\{-I_4, I_4\}$,

$$\begin{aligned}
x_2 &= \mathrm{diag}\{-I_2, I_2, -I_2, I_2\},\\
x_3 &= \mathrm{diag}\{-1, 1, -1, 1, -1, 1, -1, 1\}.
\end{aligned}$$

For each $r \leq 3$, let $F'_r = \langle \sigma_2, \omega_0, x_1, \ldots, x_r \rangle$.

Proposition 7.14 *For an elementary abelian 2-group $F \subset G$, if F contains an element conjugate to σ_3 but contains no elements conjugate to σ_2, then F is conjugate to one of* $\{F''_{r,s} : r + s \leq 3\}$, $\{F'_r : r \leq 3\}$.

Proof We may and do assume that $\sigma_3 \in F$, then

$$F \subset G^{\sigma_3} \cong (\mathrm{SU}(8)/\langle iI \rangle) \rtimes \langle \omega_0 \rangle.$$

By Lemma 7.3, we have $\mathrm{rank}(F/H_F) \leq 2$.

When $\mathrm{rank}(F/H_F) = 1$, $F \subset (G^{\sigma_3})_0 \cong \mathrm{SU}(8)/\langle iI \rangle$. As F has no elements conjugate to σ_2, so any element of F is conjugate to τ_2 or $\sigma_3\tau_2$ in $\mathrm{SU}(8)/\langle iI \rangle$, where $\tau_2 = \begin{pmatrix} -I_4 & \\ & I_4 \end{pmatrix}$. Then $F \sim F''_{r,s}$ for some $r, s \geq 0$ with $r + s \leq 3$ by Proposition 2.24.

When $\mathrm{rank}(F/H_F) = 2$, we may and do assume that $\omega_0 \in F$ as well, then

$$F \subset (G^{\sigma_3})^{\omega_0} = (\mathrm{SO}(8)/\langle -I \rangle) \times \langle \sigma_3, \omega_0 \rangle.$$

We have $H_F = F \cap \mathrm{SO}(8)/\langle -I\rangle$. Since F contains no elements conjugate to σ_2, so any involution in H_F is conjugate to $\eta_2 = \mathrm{diag}\{-I_4, I_4\}$ in $\mathrm{SO}(8)/\langle -I\rangle$. Then $F \sim F'_r$ for some $r \leq 3$ by Proposition 2.24. □

Proposition 7.15 *We have* $\mathrm{rank} A_{F''_{r,s}} = \mathrm{rank} A_{F'_r} = r$ *and any two groups in* $\{F''_{r,s} : r+s \leq 3\}$, $\{F'_r : r \leq 3\}$ *are non-conjugate.*

Proof By Lemma 7.13, we get $\mathrm{rank} A_{F''_{r,s}} = \mathrm{rank} A_{F'_r} = r$. Then the conjugacy class of any group F in $\{F''_{r,s}\}$, $\{F'_r\}$ is determined by the numbers

$$(\mathrm{rank}(F/H_F), \mathrm{rank} A_F, \mathrm{rank} F).$$

□

7.3 Pure σ_1 subgroups

A subgroup F of G is called a *pure* σ_1 *subgroup* if any of its non-identity element is conjugate to σ_1.

By Table 2, we have $G^{\sigma_1} \cong (\mathrm{Spin}(12) \times \mathrm{Sp}(1))/\langle (c, 1), (-c, -1)\rangle$, where $c = e_1 e_2 \ldots e_{12}$. From [8, Page 16], we see that $(e_1e_2e_3e_4, 1)$, $(e_1e_2, \mathbf{i})$, $(e_1e_2e_3e_4e_5e_6, \mathbf{i})$, $(\Pi, 1)$, $(\Pi, -1)$ represent the conjugacy classes of involutions in G^{σ_1} except $\sigma_1 = (1, -1)$ and we have the following conjugacy classes in G,

$$\begin{aligned}
&(e_1e_2e_3e_4, 1) \sim \sigma_1,\\
&(e_1\Pi e_1, i) \sim \sigma_1,\ (e_1e_2, \mathbf{i}) \sim \sigma_2,\\
&(e_1e_2e_3e_4e_5e_6, \mathbf{i}) \sim \sigma_3,\\
&(\Pi, 1) \sim \sigma_2,\ (\Pi, -1) \sim \sigma_3.
\end{aligned}$$

Here

$$\Pi = \frac{1+e_1e_2}{\sqrt{2}} \frac{1+e_3e_4}{\sqrt{2}} \frac{1+e_5e_6}{\sqrt{2}} \frac{1+e_7e_8}{\sqrt{2}} \frac{1+e_9e_{10}}{\sqrt{2}} \frac{1+e_{11}e_{12}}{\sqrt{2}} \in \mathrm{Spin}(12).$$

Let

$$\begin{aligned}
K_1 &= \langle (e_1\Pi e_1^{-1}, \mathbf{i}), (e_1\Pi' e_1^{-1}, \mathbf{j})\rangle,\\
K_2 &= \langle (e_1e_2e_3e_4, 1), (e_5e_6e_7e_8, 1)\rangle,\\
K_3 &= \langle (e_1\Pi e_1^{-1}, \mathbf{i}), (-e_1e_2e_3e_4, 1)\rangle,\\
K_4 &= \langle (e_1\Pi e_1^{-1}, \mathbf{i}), (e_1e_2e_3e_4, 1)\rangle,\\
K_5 &= \langle (e_1e_2e_3e_4, 1), (e_1e_2e_5e_6, 1)\rangle,
\end{aligned}$$

where

$$\Pi' = \frac{1+e_1e_3}{\sqrt{2}} \frac{1+e_4e_2}{\sqrt{2}} \frac{1+e_5e_7}{\sqrt{2}} \frac{1+e_8e_6}{\sqrt{2}} \frac{1+e_9e_{11}}{\sqrt{2}} \frac{1+e_{12}e_{10}}{\sqrt{2}}.$$

Lemma 7.16 *We have* $\Pi^2 = \Pi'^2 = [\Pi, \Pi'] = c$.

Proof $\Pi^2 = \Pi'^2 = c$ is clear. Calculation shows that

$$\Pi\Pi' = \frac{1+e_1e_4}{\sqrt{2}} \frac{1+e_2e_3}{\sqrt{2}} \frac{1+e_5e_8}{\sqrt{2}} \frac{1+e_6e_7}{\sqrt{2}} \frac{1+e_9e_{12}}{\sqrt{2}} \frac{1+e_{10}e_{11}}{\sqrt{2}},$$

so $(\Pi\Pi')^2 = c$. Then

$$[\Pi, \Pi'] = \Pi\Pi'\Pi^{-1}\Pi'^{-1} = \Pi\Pi'(c\Pi)(c\Pi') = (\Pi\Pi')^2 = c.$$

□

Lemma 7.17 *In G, we have $K_1 \sim K_3 \sim K_5 \sim \Gamma_1$ and $K_2 \sim K_4 \sim \Gamma_2$.*

Proof Since $(\mathfrak{u}_0^{\sigma_1})^{K_1} \cong \mathfrak{sp}(3)$ is not a symmetric subgroup of $\mathfrak{u}_0^{\Gamma_2} \cong \mathfrak{so}(8) \oplus (\mathfrak{sp}(1))^2$ and $(\mathfrak{e}_7^{\sigma_1})^{K_2} \cong (\mathfrak{sp}(1))^7$ is not a symmetric subgroup of $\mathfrak{u}_0^{\Gamma_1} \cong \mathfrak{su}(6) \oplus (i\mathbb{R})^2$, we get that $K_1 \sim \Gamma_1$ and $K_2 \sim \Gamma_2$.

Choose a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{e}_7$, we may assume that $\sigma_1 = \exp(\pi i H_2')$. Then $\mathfrak{g}^{\sigma_1}$ has a simple root system

$$\{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \beta, \alpha_7\}(\text{Type D}_6) \bigsqcup \{\alpha_1\},$$

where $\beta = \alpha_1 + 2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5$. By identifying conjugacy (classes of) elements in $\exp(\mathfrak{h}_0)$ and in $\mathrm{Spin}(12) \times \mathrm{Sp}(1)/\langle(c, 1), (-c, -1)\rangle$, we get the conjugacy relations

$$(\exp(\pi i H_1'), \exp(\pi i H_3'), \exp(\pi i H_2')) \sim \left(\sigma_1, (e_1\Pi e_1^{-1}, \mathbf{i}), e_1e_2e_3e_4\right)$$

and

$$(\exp(\pi i H_1'), \exp(\pi i H_2'), \exp(\pi i H_4')) \sim (\sigma_1, e_1e_2e_3e_4, e_1e_2e_5e_6).$$

Then we have $K_3 \sim K_5 \sim \Gamma_1$ and $K_4 \sim \Gamma_2$. □

Let $\pi : \mathrm{Spin}(12) \longrightarrow \mathrm{SO}(12)$ be the natural projection.

Lemma 7.18 *In $\mathrm{Spin}(12)$, we have $\Pi \sim \Pi^{-1}$, $\Pi \not\sim -\Pi$ and $\Pi \not\sim \pm e_1\Pi e_1^{-1}$.*

Proof We have $(e_1e_3e_5e_7e_9e_{11})\Pi(e_1e_3e_5e_7e_9e_{11})^{-1} = \Pi^{-1}$, so $\Pi \sim \Pi^{-1}$. Since

$$\mathrm{SO}(12)^{\pi(\Pi)} = \{g \in \mathrm{Spin}(12) | g\Pi g^{-1} = \pm\Pi\}/\langle -1\rangle,$$

$-1 \in \{g \in \mathrm{Spin}(12) | g\Pi g^{-1} = \Pi\}$ and $\mathrm{SO}(12)^{\pi(\Pi)} = \mathrm{U}(6)$ is connected, so we must have $\Pi \not\sim -\Pi$. We have $\pi(\Pi) = J_6 \in \mathrm{SO}(12)$ and $\pi(\pm e_1\Pi e_1^{-1}) = I_{1,11}J_6I_{1,11}^{-1}$. Since $J_6 \not\sim_{\mathrm{SO}(12)} I_{1,11}J_6I_{1,11}^{-1}$, so $\Pi \not\sim_{\mathrm{Spin}(12)} \pm e_1\Pi e_1^{-1}$. □

Lemma 7.19 *We have $\mathrm{Aut}(\mathfrak{e}_7)^{\Gamma_1} = (\mathrm{Aut}(\mathfrak{e}_7)^{\Gamma_1})_0 \rtimes \langle(e_1\Pi' e_1, \mathbf{j})\rangle$ and*

$$(\mathrm{Aut}(\mathfrak{e}_7)^{\Gamma_1})_0 \cong (\mathrm{SU}(6) \times \mathrm{U}(1) \times \mathrm{U}(1))/\langle(\omega I, \omega^{-1}, 1), (-I, 1, 1)\rangle.$$

Proof First we calculate $\mathrm{Spin}(12)^{\Pi}$. We have

$$\mathrm{SO}(12)^{\pi(\Pi)} \cong \mathrm{U}(6) = (\mathrm{SU}(6) \times \mathrm{U}(1))/\langle \eta I, \eta^{-1}\rangle,$$

where $\eta = e^{\frac{2\pi i}{6}}$. Then $\mathrm{Spin}(12)^{\Pi} = (\mathrm{SU}(6) \times A)/Z$, where

$$A = \{\prod_{1 \leq jleq 6} (\cos\theta + \sin\theta e_{2j-1}e_{2j}) : \theta \in \mathbb{R}\} \cong \mathrm{U}(1)$$

and $Z \subset Z(\mathrm{SU}(6)) \times A$. The isomorphism $\mathrm{U}(1) \cong A$ maps $-1 \in \mathrm{U}(1)$ to $c \in A$, and $\pi(c) = -I \in \mathrm{SO}(12)$, so

$$\pi : \mathrm{Spin}(12)^{\Pi} \longrightarrow \mathrm{SO}(12)^{\pi(\Pi)}$$

is an isomorphism when it is restricted to $\mathrm{SU}(12)$ or A.

We show that $-c \in \mathrm{SU}(6) \subset \mathrm{Spin}(12)^{\Pi}$. For this, we first look at the case of $n = 4$. For

$$\Pi_0 = \frac{1 + e_1e_2}{\sqrt{2}} \frac{1 + e_3e_4}{\sqrt{2}} \in \mathrm{Spin}(4),$$

we have $\Pi_0^2 = c_0 = e_1e_2e_3e_4$. We have an isomorphism

$$\mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1),$$

which maps $-1 \in \mathrm{Spin}(4)$ to $(-1, -1) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ and maps $c_0 \in \mathrm{Spin}(4)$ to $(-1, 1) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. Then $\Pi \in \mathrm{Spin}(4)$ is mapped to $(\mathbf{i}, 1)$ or $(\mathbf{i}, -1)$ in $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$. Since

$$(\mathrm{Sp}(1) \times \mathrm{Sp}(1))^{(\mathbf{i},\pm 1)} = \mathrm{U}(1) \times \mathrm{Sp}(1),$$

so $(1, -1)$ is in the semisimple part $\mathrm{Sp}(1)$ of it. Then $-c_0 \in \mathrm{Spin}(4)$ is in the $\mathrm{SU}(2)$ part of $\mathrm{Spin}(4)^{\Pi}$.

As Π is in block form, so $-c \in \mathrm{SU}(6) \subset \mathrm{Spin}(12)^{\Pi}$ as well. Since $(-c)c = -1 \neq 1 \in \mathrm{Spin}(12)$, $\pi(-1) = 1$, and π is a 2-fold covering, so

$$\mathrm{Spin}(12)^{\Pi} = (\mathrm{SU}(6) \times \mathrm{U}(1))/\langle(\omega I, \omega^{-1})\rangle$$

(here we identify A and $\mathrm{U}(1)$). By Lemma 7.18 and Steinberg 's theorem, we get that

$$\mathrm{Aut}(\mathfrak{e}_7)^{\Gamma_1} = (\mathrm{Aut}(\mathfrak{e}_7)^{\Gamma_1})_0 \rtimes \langle(e_1\Pi' e_1, \mathbf{j})\rangle.$$

The description of $(\mathrm{Aut}(\mathfrak{e}_7)^{\Gamma_1})_0$ follows from the description of $\mathrm{Spin}(12)^{\Pi}$ as above. □

In $G^{\sigma_1} \cong \mathrm{Spin}(12) \times \mathrm{Sp}(1)/\langle(c, 1), (-c, -1)\rangle$, let

$$H_1 = \left\langle \sigma_1, \left(e_1\Pi e_1^{-1}, \mathbf{i}\right), \left(e_1\Pi' e_1^{-1}, \mathbf{j}\right)\right\rangle,$$
$$H_2 = \langle\sigma_1, (e_1e_2e_3e_4, 1), (e_5e_6e_7e_8, 1)\rangle$$

and $H_3 = \langle\sigma_1, (e_1\Pi e_1^{-1}, \mathbf{i}), (e_1e_2e_3e_4, 1)\rangle$. Then any Klein four subgroup of H_1 is conjugate to Γ_1; any Klein four subgroup of H_2 is conjugate to Γ_2; a Klein four subgroup of H_3 is conjugate to Γ_2 if and only if it contains $(e_1e_2e_3e_4, 1)$, otherwise it is conjugate to Γ_1.

Lemma 7.20 *We have* $G^{H_1} = (\mathrm{Sp}(3)/\langle -I\rangle) \times H_1$ *and the involutions* $I_{1,2}$, $\mathbf{i}I$ *of* $\mathrm{Sp}(3)/\langle -I\rangle$ *are conjugate to* σ_1, σ_2 *in* $\mathrm{Aut}(\mathfrak{e}_7)$ *respectively.*

Proof $G^{H_1} = (\mathrm{Sp}(3)/\langle -I\rangle) \times H_1$ follows from Lemma 7.19 and the fact

$$\mathfrak{su}(6)^{e_1\Pi' e_1^{-1}} = \mathfrak{sp}(3).$$

A little more calculation by following the chain $\mathrm{Sp}(3) \subset \mathrm{SU}(6) \subset \mathrm{SO}(12)$ shows that $I_{1,2}$, $\mathbf{i}I \in \mathrm{Sp}(3)$ are conjugate to $e_1e_2e_3e_4$, Π in $\mathrm{Spin}(12)$ respectively. Then they are conjugate to σ_1, σ_2 in $\mathrm{Aut}(\mathfrak{e}_7)$ respectively. □

Lemma 7.21 *Any rank* 3 *elementary abelian 2- pure* σ_1 *subgroup* F *of* G *is conjugate to one of* H_1, H_2, H_3.

Proof For a rank 3 pure σ_1 elementary abelian 2-subgroup F of G, we may and do assume that $\sigma_1 \in F$. Then

$$F \subset G^{\sigma_1} \cong \mathrm{Spin}(12) \times \mathrm{Sp}(1)/\langle(c, 1), (-c, -1)\rangle$$

and any element of $F - \{1, \sigma_1\}$ is conjugate to $(e_1\Pi e_1^{-1}, \mathbf{i})$ or $(e_1e_2e_3e_4, 1)$ in G^{σ_1}.

When any Klein four subgroup of F is conjugate to Γ_1, we have $F \sim H_1$ by Lemma 7.17; when any Klein four subgroup of F is conjugate to Γ_2, similarly we have $F \sim H_2$ by Lemma 7.17. For the remaining cases, it is clear that $F \sim H_3$. □

We have defined the subgroups $\{F_{r,s} : r \leq 2, s \leq 3\}$ and $\{F''_{r,s} : r+s \leq 3\}$ in the last two subsections. The subgroup $F_{r,s}$ contains a Klein four subgroup conjugate to F_6; $F''_{r,s}$ does not contain any element conjugate to σ_2 and we have $\mathrm{rank}(F''_{r,s}/H_{F''_{r,s}}) = 1$. For any (r, s) with $r+s \leq 3$, let

$$F'''_{r,s} = H_{F''_{r,s}} = \{1\} \cup \{x \in F''_{r,s} | x \sim \sigma_1\};$$

for any $r \leq 2$, let

$$F''_r = H_{F_{r,3}} = \{1\} \cup \{x \in F_{r,3} | x \sim \sigma_1\}.$$

Proposition 7.22 *Any pure σ_1 elementary abelian 2-group $F \subset G$ is conjugate to F''_{r+3} for some $r \leq 2$ or $F'''_{r,s}$ for some (r, s) with $r+s \leq 3$.*

Proof When F contains a subgroup conjugate to H_1, we may and do assume that $H_1 \subset F$, then

$$F \subset G^{H_1} = (G^{\sigma_1})^{(e_1\Pi e_1,\mathbf{i}),(e_1\Pi' e_1,\mathbf{j})} \cong (\mathrm{Sp}(3)/\langle -I\rangle) \times \langle \sigma_1, (e_1\Pi e_1, \mathbf{i}), (e_1\Pi' e_1, \mathbf{j})\rangle.$$

Since F is pure σ_1, by Lemma 7.20 we have any non-identity element of $F \cap (\mathrm{Sp}(3)/\langle -I\rangle)$ is conjugate to $I_{1,2}$ in $\mathrm{Sp}(3)/\langle -I\rangle$. Then $F \cap (\mathrm{Sp}(3)/\langle -I\rangle)$ is conjugate to a subgroup of $\langle I_{2,1}, I_{1,2}\rangle$, which is a subgroup of $\langle \mathbf{i}I, \mathbf{j}I, I_{2,1}, I_{1,2}\rangle$. We may and do assume that $\mathbf{i}I, \mathbf{j}I \in C_G(F)$. Since Non-identity elements of $\langle \mathbf{i}I, \mathbf{j}I\rangle$ are all conjugate to σ_2 in G, so $\langle F, \mathbf{i}I, \mathbf{j}I\rangle$ is conjugate to some $F_{r,s}$ (cf. Proposition 7.7). Then F is conjugate to some $H_{F_{r,s}} = \{1\} \cup \{x \in F_{r,s} | x \sim \sigma_1\}$. Since we assume that $H_1 \subset F$, so we have $s = 3$. Then F is conjugate to F''_r.

If F does not contain any subgroup conjugate to H_1 but contains a subgroup conjugate to Γ_1, we may an do assume that $\sigma_1, (e_1\Pi e_1^{-1}, \mathbf{i}) \in F$. Since F does not contain any subgroup conjugate to H_1, so

$$F \subset \left((G^{\sigma_1})^{(e_1\Pi e_1^{-1},\mathbf{i})}\right)_0 \cong (\mathrm{SU}(6) \times \mathrm{U}(1) \times \mathrm{U}(1))/\left\langle \left(e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}}, 1\right), (-I, 1, 1)\right\rangle.$$

Since F is pure σ_1, we have

$$F = (F \cap (\mathrm{SU}(6)/\langle -I\rangle)) \times \left\langle \sigma_1, \left(e_1\Pi e_1^{-1}, \mathbf{i}\right)\right\rangle$$

and any element in $F \cap (\mathrm{SU}(6)/\langle -I\rangle)$ is conjugate to $I_{2,4}$. Then F is toral (cf. Proposition 2.4).

If F does not contain any subgroup conjugate to Γ_1, then any Klein four subgroup of F is conjugate to Γ_2. When $\mathrm{rank}(F) \geq 3$, we may an do assume that $H_2 \subset F$. Since there are no elements $x \in (G^{\sigma_1})^{H_2} - H_2$ such that any Klein four subgroup of $\langle x, H_2\rangle$ is conjugate to Γ_2, so $\mathrm{rank}(F) \leq 3$. Then F is conjugate to one of $1, \langle\sigma_1\rangle, \Gamma_2, H_2$, so F is toral.

For a toral and pure σ_1 elementary abelian 2-subgroup F of G, there exists a Cartan subalgebra $\mathfrak{h}_0$ such that $F \subset \exp(\mathfrak{h}_0)$. Choose a Chevelley involution θ of $\mathfrak{e}_7$ with respect to $\mathfrak{h}_0$. Then $F' = \langle F, \theta\rangle$ satisfies $\mathrm{Res}(F'/H_{F'}) = 1$ and any involution in $F' - H_{F'}$ is conjugate to σ_3. Then F' is conjugate to $F''_{r,s}$ for some (r, s) with $r+s \leq 3$. Then F is conjugate to $F'''_{r,s}$. □

Proposition 7.23 *For any $r+s \leq 3$, we have $\mathrm{rank} A_{F'''_{r,s}} = r$; for any $r \leq 2$, we have $\mathrm{rank} A_{F''_r} = r$.*

Any two subgroups in $\{F'''_{r,s} : r+s \leq 3\}, \{F''_r : r \leq 2\}$ are non-conjugate.

Proof By Propositions 7.9 and 7.14, we get $\mathrm{rank} A_{F'''_{r,s}} = r$ and $\mathrm{rank} A_{F''_r} = r$. Then any two groups in $\{F'''_{r,s} : r+s \leq 3\}, \{F''_r : r \leq 2\}$ are non-conjugate. □

7.4 Automizer groups and inclusion relations

Corollary 7.24 *G has 78 conjugacy classes of elementary abelian 2-subgroups.*

Proof By Propositions 7.7, 7.10, 7.14, 7.15, 7.22 and7.23,we get that G has

$$3\times 4+3\times 4+3\times 6+3\times 3+10+4+10+3=78$$

conjugacy classes of elementary abelian 2-subgroups. □

Proposition 7.25 *For an isomorphism $f: F \longrightarrow F'$ between two elementary abelian 2-subgroups of G, if $f(x)\sim x$ for any $x\in F$ and $m_{F'}(f(x),f(y))=m_F(x,y)$ for any $x,y\in H_F$, then $f=\mathrm{Ad}(g)$ for some $g\in G$.*

Proof When F contains an element conjugate to σ_2, we may and do assume that $\sigma_2\in F\cap F'$ and $f(\sigma_2)=\sigma_2$, then

$$F,F'\subset G^{\sigma_2}\cong\langle\omega\rangle\rtimes((\mathrm{E}_6\times\mathrm{U}(1))/\langle(c,e^{\frac{2\pi i}{3}})).$$

From the description of conjugacy classes of elements in G^{σ_2}, we get that $f(x)\sim_{G^{\sigma_2}}x$ for any $x\in F$ by the assumption in the proposition. Then $f=\mathrm{Ad}(g)$ for some $g\in G^{\sigma_2}$ by Proposition 6.9.

When $\mathrm{rank}(F/H_F)=1$ and F contains no elements conjugate to σ_2, we may and do assume that $\sigma_3\in F\cap F'$ and $f(\sigma_3)=\sigma_3$, then

$$F,F'\subset G^{\sigma_3}\cong\langle\omega_0\rangle\rtimes(\mathrm{SU}(8)/\langle iI\rangle)$$

and any element in $(H_F\cup H_{F'})-\{1\}$ is conjugate to $I_{4,4}$ in $\mathrm{SU}(8)/\langle iI\rangle$. Since the functions m_F on $H_F\times H_F$ and $m_{F'}$ on $H_{F'}\times H_{F'}$ are identical to the anti-symmetric bilinear form when H_F, $H_{F'}$ are regarded as subgroups of $\mathrm{PU}(8)$ (cf. Lemma 7.13). Then $f=\mathrm{Ad}(g)$ f or some $g\in G^{\sigma_3}$ by Proposition 2.24.

When $\mathrm{rank}(F/H_F)=2$ and F contains no elements conjugate to σ_2, we may and do assume that $\sigma_3,\omega_0\in F$, then

$$F,F'\subset(G^{\sigma_3})^{\omega_0}\cong\mathrm{SO}(8)/\langle -I\rangle$$

and any element in $(H_F\cup H_{F'})-\{1\}$ is conjugate to $I_{4,4}$ in $\mathrm{SO}(8)/\langle -I\rangle$. Then $f=\mathrm{Ad}(g)$ for some $g\in(G^{\sigma_3})^{\omega_0}$ by Proposition 2.24.

When F is pure σ_1, we get the conclusion by the considering the preserving of $m_F, m_{F'}$ under f. □

Proposition 7.26 *We have the following description for the automizer groups,*

(1) *for* $r\leq 2,\ s\leq 3,\ W(F_{r,s})\cong\mathrm{Hom}(\mathbb{F}_2^2,\mathbb{F}_2^r)\rtimes(\mathrm{GL}(2,\mathbb{F}_2)\times P(r,s,\mathbb{F}_2))$;
(2) *for* $r\leq 2,\ s\leq 3,\ W(F'_{r,s})\cong\mathbb{F}_2^r\rtimes P(r,s,\mathbb{F}_2)$;
(3) *for* $\epsilon+\delta\leq 1,\ r+s\leq 2$,

$$W(F_{\epsilon,\delta,r,s})=(\mathbb{F}_2^{r+2s+\epsilon+2\delta+1}\rtimes\mathrm{Hom}(\mathbb{F}_2^{\epsilon+2\delta+2s+1},\mathbb{F}_2^r))\rtimes(\mathrm{GL}(r,\mathbb{F}_2)\times\mathrm{Sp}(\delta+s;\epsilon)).$$

(4) *for* $\epsilon+\delta\leq 1,\ r+s\leq 2$,

$$W(F'_{\epsilon,\delta,r,s})=\mathrm{Hom}(\mathbb{F}_2^{\epsilon+2\delta+2s+1},\mathbb{F}_2^r)\rtimes(\mathrm{GL}(r,\mathbb{F}_2)\times\mathrm{Sp}(\delta+s;\epsilon)).$$

(5) *for* $r+s\leq 3,\ W(F''_{r,s})\cong(\mathbb{F}_2^{r+2s}\rtimes\mathrm{Hom}(\mathbb{F}_2^{2s},\mathbb{F}_2^r))\rtimes(\mathrm{GL}(r,\mathbb{F}_2)\times\mathrm{Sp}(s))$;
(6) *for* $r\leq 3,\ W(F'_r)\cong\mathrm{Hom}(\mathbb{F}_2^2,\mathbb{F}_2^r)\rtimes(\mathrm{GL}(r,\mathbb{F}_2)\times\mathrm{GL}(2,\mathbb{F}_2))$;
(7) *for* $r+s\leq 3,\ W(F'''_{r,s})\cong\mathrm{Hom}(\mathbb{F}_2^{2s},\mathbb{F}_2^r)\rtimes(\mathrm{GL}(r,\mathbb{F}_2)\times\mathrm{Sp}(s))$;
(8) *for* $r\leq 2,\ W(F''_r)\cong P(r,3,\mathbb{F}_2)$.

Proof By Proposition 7.25, we need to find all automorphisms of F preserving the conjugacy classes of involutions and the form m on H_F.

We prove (4). Let $F = F'_{\epsilon,\delta,r,s}$. Then F has a decomposition $F = A_F \times F'$ with $A_F = \mathbb{F}_2^r$ be the translation subgroup and $F' \sim F'_{\epsilon,\delta,0,s}$. By Proposition 7.25, we have

$$W(F) \cong \mathrm{Hom}(F', A_F) \rtimes (\mathrm{GL}(r, \mathbb{F}_2) \times W(F')).$$

So we only need to prove in the case of $r = 0$. Assume that $r = 0$ from now on.

Any element in $W(F)$ preserves the symplectic form m on H_F. Since $\mathrm{rank}(\ker m) = \epsilon$, so we have a homomorphism

$$p : W(F) \longrightarrow \mathrm{Sp}(\delta + s; \epsilon).$$

We show that this homomorphism is an isomorphism, which finishes the proof.

For any $f : F \longrightarrow F$ with $f|_{H_F} = 1$, since $F = H_F \rtimes \langle z \rangle$ with $z \sim \sigma_2$, let $f(z) = zx_0$ for some $x_0 \in H_F$. The for any $x \in H_F$, $f(zx) = zxx_0$, so $zx \sim zxx_0$. This just said $x_0 \in A_F$. Since we assume that $r = 0$ (equivalent to $A_F = 1$), so $x_0 = 1$. And so $f = \mathrm{id}$. Then p is injective.

By Proposition 7.25, $W(F)$ permutes transitively elements of F conjugate to σ_2. There are

$$\frac{2^{2\delta+2s+\epsilon} + (1-\epsilon)(-1)^{\delta}2^{\epsilon+\delta+s}}{2} = 2^{s+\delta-1}(2^{s+\delta+\epsilon} + (1-\epsilon)(-1)^{\delta}2^{\epsilon})$$

such elements. It is clear that the stabilizer of $W(F)$ at z is $\mathrm{Sp}(s; \epsilon, \delta)$. So

$$|W(F)| = |\mathrm{Sp}(s; \epsilon, \delta)| 2^{s+\delta-1}(2^{s+\delta+\epsilon} + (1-\epsilon)(-1)^{\delta}2^{\epsilon}).$$

By Propositions 2.32 and 2.33, this is also equal to $|\mathrm{Sp}(s + \delta; \epsilon)|$. Then p is surjective.

(3) follows from (4) immediately.

The proof for the other cases easier, we use the facts that $\mathrm{rank} A_F = r$ and the form m on H_F/A_F is non-degenerate. □

Remark 7.27 We have the following containment relations,

$$F'''_{\epsilon+r,\delta+s} \subset F'_{\epsilon,\delta,r,s}, \quad F'''_{\epsilon+r,\delta+s} \subset F''_{\epsilon+r,\delta+s}, \quad F''_{\epsilon+r,\delta+s} \subset F_{\epsilon,\delta,r,s},$$
$$F'_{r+s+\delta} \subset F_{\epsilon,\delta,r,s}, \quad F''_{r+3} \subset F'_{r,3},$$

together with those obvious relations, they consist in all containment relations (in the sense of conjugacy) between these subgroups

8 E_8

Let $G = \mathrm{Aut}(\mathfrak{e}_8)$. By Table 2, G has two conjugacy classes of involutions with representatives σ_1, σ_2 and we have

$$G^{\sigma_1} \cong (\mathrm{E}_7 \times \mathrm{Sp}(1))/\langle (c, -1) \rangle,$$
$$G^{\sigma_2} \cong \mathrm{Spin}(16)/\langle c' \rangle,$$

where c is the unique non-trivial central element of E_7 and $c' = e_1 e_2 \dots e_{16} \in \mathrm{Spin}(16)$.

In $G^{\sigma_1} \cong (\mathrm{E}_7 \times \mathrm{Sp}(1))/\langle (c, -1) \rangle$, let $\eta_1, \eta_2 \in \mathrm{E}_7$ be involutions such that there exists Klein four groups $F, F' \subset \mathrm{E}_7$ with non-identity elements all conjugate to η_1 or η_2 respectively and

$$\mathfrak{e}_7^F \cong \mathfrak{su}(6) \oplus (i\mathbb{R})^2, \ \mathfrak{e}_7^{F'} \cong \mathfrak{so}(8) \oplus (\mathfrak{sp}(1))^3.$$

Then $c\eta_1 \sim_{E_7} \eta_2$, $c\eta_2 \sim_{E_7} \eta_1$. Let $\tau_1 = (\eta_1, 1), \tau_2 = (\eta_2, 1) \in G^{\sigma_1}$. Let $\eta_3, \eta_4 \in \mathrm{E}_7$ be involutions with $\eta_3^2 = \eta_4^2 = c$ and

$$\mathfrak{e}_7^{\eta_3} \cong \mathfrak{e}_6 \oplus i\mathbb{R}, \ \mathfrak{e}_7^{\eta_4} \cong \mathfrak{su}(8).$$

Then $c\eta_3 \sim_{\mathrm{E}_7} \eta_3$, $c\eta_4 \sim_{\mathrm{E}_7} \eta_4$. Let $\tau_3 = (\eta_3, \mathbf{i})$, $\tau_4 = (\eta_4, \mathbf{i})$. By [8, Page 17], we see that $\tau_1, \tau_2, \tau_3, \tau_4$ represent all conjugacy classes of involutions in G^{σ_1} except σ_1 and we have the following conjugacy classes in G,

$$\tau_1 \sim \sigma_1, \ \tau_2 \sim \sigma_2,$$
$$\tau_3 \sim \sigma_1, \ \tau_4 \sim \sigma_2$$

In $G^{\sigma_2} \cong \mathrm{Spin}(16)/\langle c \rangle$, let

$$\tau_1 = e_1e_2e_3e_4, \ \tau_2 = e_1e_2e_3\ldots e_8,$$
$$\tau_3 = \Pi, \ \tau_4 = -\Pi,$$

where

$$\Pi = \frac{1+e_1e_2}{\sqrt{2}}\frac{1+e_3e_4}{\sqrt{2}}\cdots\frac{1+e_{15}e_{16}}{\sqrt{2}}.$$

By [8, Page 17], we see that $\tau_1, \tau_2, \tau_3, \tau_4$ represent all conjugacy classes of involutions in G^{σ_2} except σ_2 and we have the following conjugacy classes in G,

$$\tau_1 \sim \tau_3 \sim \sigma_1,$$
$$\tau_2 \sim \tau_4 \sim \sigma_2.$$

Moreover in G^{σ_2}, we have

$$\sigma_2\tau_1 \sim_{G^{\sigma_2}} \tau_1, \ \sigma_2\tau_2 \sim_{G^{\sigma_2}} \tau_2,$$
$$\sigma_2\tau_3 \sim_{G^{\sigma_2}} \tau_4, \ \sigma_2\tau_4 \sim_{G^{\sigma_2}} \tau_3.$$

These are obtained from calcualtions in $\mathrm{Spin}(16)/\langle c \rangle$.

Definition 8.1 Let F be an elementary abelian 2-subgroup of G. For any $x \in F$ with $x \sim \sigma_1$, let

$$H_x = \{y \in F | xy \not\sim y\}.$$

Let

$$H_F := \langle \{H_x : x \in F, x \sim \sigma_1\} \rangle = \langle \{x : x \in F, x \sim \sigma_1\} \rangle.$$

Lemma 8.2 *Let F be an elementary abelian 2-subgroup of G. For any x with $x \sim \sigma_1$, H_x is a subgroup and* $\mathrm{rank}(F/H_x) \leq 2$.

Proof We may and do assume that $x = \sigma_1$, then

$$F \subset G^{\sigma_1} \cong \mathrm{E}_7 \times \mathrm{Sp}(1)/\langle (c, -1) \rangle.$$

For an element $y \in F \subset G^{\sigma_1}$ with $y^2 = 1$, $\sigma_1 y \not\sim y$ if and only if y is conjugate to $1, \sigma_1, \tau_1, \tau_2$ in G^{σ_1}. Then it is also equivalent to $y \in \mathrm{E}_7 \subset G^{\sigma_1}$. So $H_x = F \cap \mathrm{E}_7$. And so it is a subgroup. Then $F/H_x \subset G^{\sigma_1}/\mathrm{E}_7 \cong \mathrm{Sp}(1)/\langle -1 \rangle$, so $\mathrm{rank}(F/H_x) \leq 2$. □

Definition 8.3 Let F an elementary abelian 2-subgroup of G, For any $x \in F$, define $\mu(x) = 1$ if $x \sim \sigma_2$ or $x = 1$; and $\mu(x) = -1$ if $x \sim \sigma_1$.

For any $x, y \in F$, define $m(x, y) = \mu(x)\mu(y)\mu(xy)$.

In general m is not a bilinear form.

Definition 8.4 For an elementary abelian 2-subgroup F of G, define the translation subgroup

$$A_F = \{x \in F | \mu(x) = 1 \, and \, m(x, y) = 1 \, for any \, y \in F\}$$

and the *defect index*

$$\mathrm{defe}(F) = |\{x \in F : \mu(x) = 1\}| - |\{x \in F : \mu(x) = -1\}|.$$

Definition 8.5 For an elementary abelian 2-subgroup F of G, we call $\mathrm{Res}(F) := \mathrm{rank}(F/H_F)$ the residual rank of F, and

$$\mathrm{Res}'(F) = max\{\mathrm{rank}(F/H_x) | x \in F, x \sim \sigma_1\}$$

the second residual rank of F.

Let $X = X_F = \{x \in F | x \sim \sigma_1\}$, define a graph with vertices set X by drawing an edge connecting $x, y \in X$ if and only if $xy \sim \sigma_2$. It is clear that this graph X is invariant under multiplication by elements in A_F. Let

$$\mathrm{Graph}(F) = X_F / A_F$$

be the quotient graph of the graph X_F modulo the action of A_F.

8.1 Subgroups from E_6

For an elementary abelian 2-subgroup F of G, if F contains a Klein four subgroup conjugate to Γ_1, we may and do assume that $\Gamma_1 = \langle \sigma_1, \tau_3 \rangle \subset F$. Then

$$F \subset G^{\Gamma_1} = ((\mathrm{E}_6 \times \mathrm{U}(1) \times \mathrm{U}(1)) / \langle (c, e^{\frac{2\pi i}{3}}, 1) \rangle) \rtimes \langle \omega \rangle,$$

where $\omega^2 = 1$, $(\mathfrak{e}_6 \oplus i\mathbb{R} \oplus i\mathbb{R})^{\omega} = \mathfrak{f}_4 \oplus 0 \oplus 0$ and $\Gamma_1 = \langle (1, -1, 1), (1, 1, -1) \rangle$.

Let $G_{\Gamma_1} = \mathrm{E}_6 \rtimes \langle \omega \rangle \subset G^{\Gamma_1}$. Let $\pi : G_{\Gamma_1} \longrightarrow \mathrm{Aut}(\mathfrak{e}_6)$ be the adjoint homomorphism and $p : G_{\Gamma_1} \longrightarrow G^{\Gamma_1}$ be the inclusion. For an elementary abelian 2-subgroup K of $\mathrm{Aut}(\mathfrak{e}_6)$, $p(\pi^{-1}(K)) \times \Gamma_1$ is the direct product of its (unique) Sylow 2-subgroup F and $\langle (c, 1, 1) \rangle$. Let $\{F_{r,s} : r \leq 2, s \leq 3\}$, $\{F'_{r,s} : r \leq 2, s \leq 3\}$, $\{F_{\epsilon,\delta,r,s} : \epsilon + \delta \leq 1, r + s \leq 2\}$, $\{F'_{\epsilon,\delta,r,s} : \epsilon + \delta \leq 1, r + s \leq 2, s \geq 1\}$ be elementary abelian 2-subgroups of E_8 obtained from elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{e}_6)$ with the corresponding notation in this way.

Let $\theta_1, \theta_2 \in \mathrm{E}_6$ be involutions with

$$\mathfrak{e}_6^{\theta_1} \cong \mathfrak{su}(6) \oplus \mathfrak{sp}(1), \ \mathfrak{e}_6^{\theta_2} \cong \mathfrak{so}(10) \oplus i\mathbb{R}.$$

Let $\theta_3 = \omega$, $\theta_4 \in \omega\mathrm{E}_6$ be involutions with

$$\mathfrak{e}_6^{\theta_3} \cong \mathfrak{f}_4 \oplus 0 \oplus 0, \ \mathfrak{e}_6^{\theta_4} \cong \mathfrak{sp}(4) \oplus 0 \oplus 0.$$

From [8, Pages 16–18] (for Types $\mathbf{E_6}$, $\mathbf{E_7}$, $\mathbf{E_8}$), we have

$$\theta_1 \sim \theta_3 \sim \sigma_1$$

and

$$\theta_2 \sim \theta_4 \sim \sigma_2.$$

More over we have

$$\theta_1\sigma \sim \theta_4\sigma \sim \sigma_2$$

and

$$\theta_2\sigma \sim \theta_3\sigma \sim \sigma_1,$$

for any $\sigma \in \Gamma_1 - \{1\}$.

Proposition 8.6 *For an elementary abelian 2-subgroup F of G, if F contains a Klein four subgroup conjugate to Γ_1, then F is conjugate to one of* $\{F_{r,s} : r \leq 2, s \leq 3\}$, $\{F'_{r,s} : r \leq 2, s \leq 2\}$, $\{F_{\epsilon,\delta,r,s} : \epsilon + \delta \leq 1, r + s \leq 2\}$, $\{F'_{\epsilon,\delta,r,s} : \epsilon + \delta \leq 1, r + s \leq 2, s \geq 1\}$.

Proof The proof is similar as that for Proposition 7.10. □

Remark 8.7 Note that $F'_{r,3}$ contains a rank 3 pure σ_1 subgroup. By Proposition 8.6, one can show that it is conjugate to $F_{r,2}$.

Proposition 8.8 *We have the following formulas for* $\mathrm{Res}F$, $\mathrm{Res}'F$, $\mathrm{rank}A_F$ *and* $\mathrm{defe}F$,

(1) *for* $F = F_{r,s}$, $r \leq 2$, $s \leq 3$, $(\mathrm{Res}F, \mathrm{Res}'F) = (0, 2)$, $\mathrm{rank}A_F = r$, $\mathrm{defe}F = 3 \cdot 2^{r+1}(2^s - 2)$;
(2) *for* $F = F'_{r,s}$, $r \leq 2$, $s \leq 2$, $(\mathrm{Res}F, \mathrm{Res}'F) = (0, 1)$, $\mathrm{rank}A_F = r$, $\mathrm{defe}F = 2^{r+1}(2^s - 2)$;
(3) *for* $F = F_{\epsilon,\delta,r,s}$, $\epsilon+\delta \leq 1$, $r+s \leq 2$, $(\mathrm{Res}F, \mathrm{Res}'F) = (1, 2)$, $\mathrm{rank}A_F = r$, $\mathrm{defe}F = (1-\epsilon)(-1)^{\delta+1}2^{r+s+\delta+1} + 2^{\epsilon+r+2\delta+2s}$;
(4) *for* $F = F'_{\epsilon,\delta,r,s}$, $\epsilon + \delta \leq 1$, $r + s \leq 2$, $s \geq 1$, $(\mathrm{Res}F, \mathrm{Res}'F) = (0, 1)$, $\mathrm{rank}A_F = r$, $\mathrm{defe}F = (1-\epsilon)(-1)^{\delta+1}2^{r+s+\delta+1}$.

Proof These formulas follow from the construction of these subgroups and the comparison of the conjugacy classes of involutions in G^{Γ_1} and in G. □

Proposition 8.9 *The subgroups* $\{F_{r,s} : r \leq 2, s \leq 3\}$, $\{F'_{r,s} : r \leq 2, s \leq 2\}$, $\{F_{\epsilon,\delta,r,s} : \epsilon + \delta \leq 1, r + s \leq 2\}$, $\{F'_{\epsilon,\delta,r,s} : \epsilon + \delta \leq 1, r + s \leq 2, s \geq 1\}$ *are not conjugate to each other.*

Proof The numbers $(\mathrm{Res}F, \mathrm{Res}'F, \mathrm{rank}A_F, \mathrm{rank}F, \mathrm{defe}F)$ clearly distinguish most of these c onjugacy classes except for some possible pairs $(F'_{r',s'}, F'_{\epsilon,\delta,r,s})$. Suppose that some $(F'_{r',s'}$ is conjugate to some $F'_{\epsilon,\delta,r,s})$. By the formulas in Proposition 8.8, we have $r' = r$ (by A_F), $s' - 1 = (1-\epsilon)(-1)^{\delta}$ (by the sign of $\mathrm{defe}F$) and $s' = 2s + 2\delta + \epsilon$ (by $\mathrm{rank}F/A_F$). Since $s' \leq 2$ and $s \geq 1$, the last equality implies that $\epsilon = \delta = 0$, $s = 1$ and $s' = 2$. Then the second equality implies that $s' = 1$. So we get a contradiction. □

8.2 Other subgroups

In $G^{\sigma_1} \cong (\mathrm{E}_7 \times \mathrm{Sp}(1))/\langle(c, -1)\rangle$, choose $x_1, x_2 \in \mathrm{E}_7$ with $x_1 \sim x_2 \sim x_1x_2 \sim \tau_4$, then

$$(G^{\sigma_1})^{x_1,x_2} = \mathrm{SO}(8)/\langle -I\rangle \times \langle\sigma_1, x_1, x_2\rangle.$$

Let $z_1 = \mathrm{diag}\{-I_4, I_4\}$,

$$z_2 = \mathrm{diag}\{-I_2, I_2, -I_2, I_2\},$$
$$z_3 = \mathrm{diag}\{-1, 1, -1, 1, -1, 1, -1, 1\}.$$

Define

$$F''_{r,s} = \langle z_1, \ldots, z_r, \sigma_1, x_1, \ldots, x_s\rangle$$

for any $r \leq 3,\ s \leq 2$,

In $G^{\sigma_2} \cong \mathrm{Spin}(16)/\langle c\rangle,\ c = e_1e_2\dots e_{16}$, let $y_1 = \sigma_1 = -1$,

$$\begin{aligned} y_2 &= e_1e_2e_3e_4e_5e_6e_7e_8, \\ y_3 &= e_1e_2e_3e_4e_9e_{10}e_{11}e_{12}, \\ y_4 &= e_1e_2e_5e_6e_9e_{10}e_{13}e_{14}, \\ y_5 &= e_1e_3e_5e_7e_9e_{11}e_{13}e_{15}. \end{aligned}$$

Define $F'_r = \langle y_1, \dots, y_r\rangle$ for any $r \leq 5$.

Lemma 8.10 *For an elementary abelian 2-subgroup F of G, if F contains no Klein four subgroup conjugate to Γ_1, but contains an element conjugate to σ_1, then*

$$\mathrm{rank} H_F/A_F = 1.$$

Proof Recall that, H_F is a subgroup of F generated by elemnts conjugate to σ_1. Let

$$Y_F = \{x \in H_F : x \sim \sigma_2\} \cup \{1\}.$$

We show that $A_F = Y_F$ under the assumption of the lemma.

Choose any $x_0 \in F$ with $x_0 \sim \sigma_1$. For any other $x \in F$ with $x \sim \sigma_1$, since F contains no Klein four subgroup conjugate to Γ_1, so $xx_0 \sim \sigma_2$. Then $x \in H_{x_0}$. By this, we get that $H_F \subset H_{x_0}$. So $H_{x_0} = H_F$ as the containment relation in the converse direction is obvious. Similarly we have $H_x = H_F$ for any $x \in F$ with $x \sim \sigma_1$. Then for any two distinct $y_1, y_2 \in Y_F$ with $y_1 \sim y_2 \sim \sigma_2$, we have $y_1y_2 \sim \sigma_2$. So Y_F is a subgroup of H_F.

Then it is clear that $Y_F = A_F$. So $\mathrm{rank} H_F/A_F = \mathrm{rank} H_F/Y_F = 1$. □

Proposition 8.11 *For an elementary abelian 2-subgroup F of G, if F contains no Klein four subgroup conjugate to Γ_1, then F is conjugate to one of $\{F''_{r,s} : r \leq 3, s \leq 2\}$, $\{F'_r : r \leq 5\}$.*

Proof When F contains no Klein four subgroup conjugate to Γ_1, but contains an element conjugate to σ_1, we may and do assume that $\sigma_1 \in F$. Then

$$F \subset G^{\sigma_1} \cong (\mathrm{E}_7 \times \mathrm{Sp}(1))/\langle (c, -1)\rangle.$$

Modulo $\mathrm{Sp}(1)$, we get a homomorphism

$$\pi : F \longrightarrow \mathrm{E}_7/\langle c\rangle \cong \mathrm{Aut}(\mathfrak{e}_7).$$

Since we assume that F contains no Klein four subgroup conjugate to Γ_1, so any element in $F - \langle\sigma_1\rangle$ is conjugate to $\tau_1 = (\eta_1, 1),\ \tau_2 = (\eta_2, 1)$ or $\tau_4 = (\eta_4, \mathbf{i})$ in $(\mathrm{E}_7 \times \mathrm{Sp}(1))/\langle(c, -1)\rangle$; and any Klein four subgroup of $F \cap \mathrm{E}_7$ has at least one element conjugate to η_2. Then $F' = \pi(F) \subset \mathrm{Aut}(\mathfrak{e}_7)$ contains no elements conjugate to η_3, and no Klein four subgroups whose fixed point subalgebra is isomorphic to $\mathfrak{su}(6) \oplus (i\mathbb{R})^2$. In the case of E_7 (Sect. 7), it corresponds to the elementary abelian 2-subgroup F' with no elements conjugate to σ_2 and the map m on $H_{F'}$ is trivial. By Propositions 7.14 and 7.22, we get that $F \sim F''_{r,s}$ for some (r, s) with $r \leq 3, s \leq 2$.

When F is pure σ_2, we may and do assume that $\sigma_2 \in F$. Then

$$F \subset G^{\sigma_2} \cong \mathrm{Spin}(16)/\langle c\rangle$$

and any element in $F - \langle\sigma_2\rangle$ is conjugate to $e_1e_2e_3e_4e_5e_6e_7e_8$ in $\mathrm{Spin}(16)/\langle c\rangle$. Then $F \sim F'_r$ for some $r \leq 5$. □

Proposition 8.12 *For any* (r, s) *with* $r \leq 3$ *and* $s \leq 2$, *we have* $\mathrm{rank} A_{F''_{r,s}} = r$; *for any* $r \leq 5$, *we have* $\mathrm{rank} A_{F'_r} = r$.

Any two subgroups in $\{F''_{r,s} : r \leq 3, s \leq 2\}, \{F'_r : r \leq 5\}$ *are non-conjugate.*

Proof The equalities $\mathrm{rank} A_{F'''_{r,s}} = r$ and $\mathrm{rank} A_{F''_r} = r$ are clear. By them, we get that any two subgroups in $\{F'''_{r,s} : r + s \leq 3\}, \{F''_r : r \leq 2\}$ are non-conjugate. □

8.3 Involution types and Automizer groups

Corollary 8.13 *G has 66 conjugacy classes of elementary abelian 2-subgroups.*

Proof By Propositions 8.6, 8.9, 8.11, 8.12, we get that G has

$$3 \times 4 + 3 \times 3 + 3 \times 6 + 3 \times 3 + 4 \times 3 + 6 = 66$$

conjugacy classes of elementary abelian 2-subgroups. □

Proposition 8.14 *For an isomorphism* $f : F \longrightarrow F'$ *between two elementary abelian 2-subgroups of* G, *if* $f(x) \sim x$ *for any* $x \in F$, *then* $f = \mathrm{Ad}(g)$ *for some* $g \in G$.

Proof When F contains a Klein four subgroup conjugate to Γ_1, this reduces to the similar statement in $\mathrm{Aut}(\mathfrak{e}_6)$ case.

When F does not contain any Klein four subgroup conjugate to Γ_1, this is already showed in the proof of Proposition 8.11. □

Definition 8.15 For an elementary abelian 2-subgroup F of G, we say that F is the orthogonal direct product of other subgroups $K_1, \ldots, K_t$ if there exists an isomorphism $f : K_1 \times \cdots \times K_t \longrightarrow F$ with

$$\mu(f(x_1, \ldots, x_t)) = \mu(x_1) \ldots \mu(x_t)$$

for any $(x_1, \ldots, x_t) \in K_1 \times \cdots \times K_t$.

Let $A = \langle \sigma_2 \rangle$. Let $B_s (s \leq 3)$ be a rank s pure σ_1 subgroup. Let $B = B_1$, $C = F_3$ and D be a rank 3 subgroup with only one element conjugate to σ_1. Then the involution types of some elementary abelian 2-subgroups of E_8 have the following description

$$\begin{aligned} &F_{r,s} = A^r \times B_s \times B_3; \; F'_{r,s} = A^r \times B_s \times B_2; \\ &F'_{\epsilon,\delta,r,s} = A^r \times C^s \times B^{\epsilon} \times B_2^{1+\delta}; \\ &F''_{r,1} = A^r \times B, \; F''_{r,2} = A_r \times C, \\ &F''_{r,3} = A^r \times D; \; F'_r = A^r \end{aligned}$$

$F_{\epsilon,\delta,r,s}$ $(s \geq 1)$ does not have a similar decomposition since elements in $F_{\epsilon,\delta,r,s} - F'_{\epsilon,\delta,r,s}$ are all conjugate to σ_2.

With the involution types available, we can describe the graphs $\mathrm{Graph}(F)$. The graphs of $F_{r,s}$ is a complete bipartite graph with $s, 3$ vertices in two parts; that of $F'_{r,s}$ is a complete bipartite graph with $s, 2$ in two parts; that of $F''_{r,s}$ $(s \geq 1)$ is a single vertex graph; that of F'_r is an empty graph. The graphs of $F_{\epsilon,\delta,r,s}$, $F'_{\epsilon,\delta,r,s}$ are not of bipartite form and a little more complicated.

In summary, we have the following statement

"*the conjugacy class of an elementary abelian 2-subgroup* $F \subset G$ *is determined by the datum* $(\mathrm{rank} F, \mathrm{rank} A_F, \mathrm{Graph}(F))$".

Proposition 8.16 *For an elementary abelian 2-subgroup $F \subset \mathrm{E}_8$, m is a bilinear form on F if and only if F is not conjugate to any of $\{F_{r,s} : r \leq 2, s \leq 3\} \cup \{F_{\epsilon,\delta,r,s} : \epsilon + \delta \leq 1, r + s \leq 2\} \cup \{F''_{r,3} : r \leq 2\}$.*

Proof When F is conjugate to one of $\{F_{r,s} : r \leq 2, s \leq 3\} \cup \{F_{\epsilon,\delta,r,s} : \epsilon+\delta \leq 1, r+s \leq 2\}$, it contains a subgroup conjugate to B_3, $F_{0,0,0,0}$ or D. The subgroups B_3, $F_{0,0,0,0}$, D contains 7, 3, 1 elements with μ-value -1 respectively, so m is not bilinear on them by Proposition 2.30.

When F is conjugate to a subgroup in the other four families, m is bilinear on F follows from the orthogonal decomposition of it. □

We can write the decomposition of involution types for some subgroups in a simpler way,

$$\begin{aligned}
&F'_{r,0} = A^r \times B_2,\\
&F'_{r,1} = A^r \times B \times B_2 = A^r \times B \times C,\\
&F'_{r,2} = A^r \times B_2 \times B_2 = A^r \times C \times C,\\
&F'_{1,0,r,s} = A^r \times C^s \times B \times B_2^1 = A^r \times B \times C^{s+1},\\
&F'_{0,\delta,r,s} = A^r \times C^s \times B_2^{1+\delta} = A^r \times B_2^{1-\delta} \times C^{s+2\delta}.
\end{aligned}$$

Proposition 8.17 (1) $r \leq 2$, $s \leq 2$, $W(F_{r,s}) \cong \mathrm{Hom}(\mathbb{F}_2^{3+s}, \mathbb{F}_2^r) \rtimes \big(\mathrm{GL}(r, F_2) \times (\mathrm{GL}(s, \mathbb{F}_2) \times \mathrm{GL}(3, \mathbb{F}_2))\big)$;

(2) $r \leq 2$, $W(F_{r,3}) \cong \mathrm{Hom}(\mathbb{F}_2^6, \mathbb{F}_2^r) \rtimes \big(\mathrm{GL}(r, \mathbb{F}_2) \times ((\mathrm{GL}(3, \mathbb{F}_2) \times \mathrm{GL}(3, \mathbb{F}_2)) \rtimes S_2)\big)$;

(3) $r \leq 2$, $s \leq 2$, $W(F'_{r,s}) \cong \mathrm{Sp}(r, s; 2s - s^2, \frac{(s-1)(s-2)}{2})$;

(4) $\epsilon+\delta \leq 1$, $r+s \leq 2$, $W(F_{\epsilon,\delta,r,s}) = \mathbb{F}_2^{r+2s+\epsilon+2\delta+2} \rtimes \mathrm{Sp}(r, s+\epsilon+2\delta; \epsilon, (1-\epsilon)(1-\delta))$;

(5) $\epsilon + \delta \leq 1$, $r + s \leq 2$, $W(F'_{\epsilon,\delta,r,s}) = \mathrm{Sp}(r, s + \epsilon + 2\delta; \epsilon, (1 - \epsilon)(1 - \delta))$.

(6) $r \leq 3$, $s \leq 2$, $W(F''_{r,s}) \cong \mathrm{Hom}(\mathbb{F}_2^s, \mathbb{F}_2^{r+1}) \rtimes \big((\mathbb{F}_2^r \rtimes \mathrm{GL}(r, \mathbb{F}_2)) \times \mathrm{GL}(s)\big)$;

(7) $r \leq 5$, $W(F'_r) \cong \mathrm{GL}(r, \mathbb{F}_2)$.

Proof By Proposition 8.14, we need to calculate automorphisms of F preserving the function μ on F.

$W(F_{r,s}) = \mathrm{Hom}(\mathbb{F}_2^{3+s}, \mathbb{F}_2^r) \rtimes W(F_{0,s})$ and $W(F_{0,s})$ stabilizes $B_s \cup B_3 \subset F_{0,s}$. By this we get (1) and (2).

When m is bilinear, (F, m, μ) is a symplectic metric space, then we can identify $W(F)$ with the automorphism group of (F, m, μ). By this we get (3) and (5).

(4) follows from (5) immediately.

For (6), we have $A_F \subset H_F \subset F$ and A_F, H_F are preserved by $W(F)$. By Lemma 8.10, we have $\mathrm{rank} A_F = r$, $\mathrm{rank} H_F/A_F = 1$, $\mathrm{rank} F/H_F = 1$, then we get the conclusion.

(7) is clear. □

We have an inclusion $p : \mathrm{E}_7 \subset \mathrm{E}_8$ since $\mathrm{E}_8^{\sigma_1} \cong (\mathrm{E}_7 \times \mathrm{Sp}(1))/\langle(c, -1)\rangle$. Let $\pi : \mathrm{E}_7 \longrightarrow \mathrm{Aut}(\mathfrak{e}_7)$ be the adjoint homomorphism, which is a 2-fold covering. For a pure σ_1 (that for E_7 case) elementary abelian 2-subgroup F of $\mathrm{Aut}(\mathfrak{e}_7)$, $p(\pi^{-1} F)$ is an elementary abelian 2-subgroup of E_8.

Proposition 8.18 *An elementary abelin 2-subgroup F of* E_8 *is conjugate to the subgroup $p(\pi^{-1}(K))$ for some pure σ_1 subgroup K of* $\mathrm{Aut}(\mathfrak{e}_7)$ *if and only if F contains an elementary x such that $x \sim \sigma_1$ and $H_x = F$.*

Proof It follows from the description of the conjugacy classes of involutions in $\mathrm{E}_8^{\sigma_1} \cong (\mathrm{E}_7 \times \mathrm{Sp}(1))/\langle(c, -1)\rangle$. □

Remark 8.19 Any subgroup of E_8 satisfying the condition in Proposition 8.18 is conjugate to one of

$$\{F_{r,1} : r \leq 2\},\ \{F'_{r,1} : r \leq 2\},\ \{F'_{1,0,r,s} : r + s \leq 2, s \geq 1\},\ \{F''_{r,1} : r \leq 3\}.$$

There are 13 such conjugacy classes in total. On the other hand, there are 13 classes of pure σ_1 subgroups of $\mathrm{Aut}(\mathfrak{e}_7)$, so for any two elementary abelian 2-subgroups $K_1,\ K_2$ of $\mathrm{Aut}(\mathfrak{e}_7)$, we have

$$p(\pi^{-1}K_1) \sim_{\mathrm{E}_8} p(\pi^{-1}K_2) \Leftrightarrow K_1 \sim_{\mathrm{Aut}(\mathfrak{e}_7)} K_2.$$

Acknowledgments The author would like to thank Professors Griess and Grodal for some helpful communications. He would like to thank Professor Jing-Song Huang for discussions on symmetric pairs and thank Professor Doran for a lot of suggestions on the mathematical writing. The author 's research is supported by a grant from Swiss National Science Foundation (Schweizerischer Nationalfonds).

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